A consideration on the one dimensional q-wavelet

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Abstract: In this paper, we give the definitions of the q-Haar and q-Gabor wavelet. Instead of using the conventional Gaussian distribution as a kernel of the Gabor wavelet, if the q-normal distribution is used, we can get the q-Gabor wavelet as a possible generalization of the Gabor wavelet. The q-normal distribution, which is given by the author, is one of the generalized Gaussian distribution. On the other hand, if two sets of the q-normal distribution are connected anti-symmetrically, we can get the q-Haar wavelet as a possible generalization of the Haar wavelet. We give experiments on the q-Gabor and q-Haar wavelet and discuss about the q-Gabor and q-Haar wavelet.

1. Introduction

In this paper, we give the definitions of the q-wavelet, especially the q-Gabor wavelet and q-Haar wavelet. Instead of using the conventional Gaussian distribution as a kernel of the Gabor wavelet, if the q-normal distribution is used, we can get the q-Gabor wavelet as a possible generalization of the Gabor wavelet. On the other hand, if two sets of the q-normal distributions are connected anti-symmetrically, we can get the q-Haar wavelet as a possible generalization of the Haar wavelet. The q-normal distribution is one of the generalized Gaussian distribution. The q-normal distribution includes the conventional Gaussian distribution as the special case ($q = 1$). The q-normal distribution gives the maximum value of the Tsallis entropy which is one of the generalized entropy and is also a non-extensive entropy. As changing only one parameter q, the q-normal distribution can realize the distribution from the uniform distribution ($q \to 3$) with non-compact support to the uniform distribution ($q \to -\infty$) with compact support which size is twice the variance continuously, through the Cauchy distribution, 't-distribution' and the conventional Gaussian distribution. For $q < 1$, the q-normal distribution has the compact support, therefore the obtained q-Gabor wavelet has the compact support. This means that we can get the orthogonal wavelet. In a following section, we give a brief review of the q-wavelet and show experiments of the q-wavelet transform.

2. The q-normal distribution

The q-normal distribution is given as,

$$p_q(x) = \frac{1}{Z_q}\left\{1 - \frac{1-q}{3-q}\frac{(x-\mu)^2}{\sigma^2}\right\}^{\frac{1}{1-q}}, \quad (1)$$

where

$$\begin{aligned} Z_q &= \int dx \left\{1 - \frac{1-q}{3-q}\frac{(x-\mu)^2}{\sigma^2}\right\}^{\frac{1}{1-q}} \\ &= \begin{cases} \left(\frac{3-q}{q-1}\sigma^2\right)^{\frac{1}{2}} B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right), & \text{for } 1 \le q < 3 \\ \left(\frac{3-q}{1-q}\sigma^2\right)^{\frac{1}{2}} B\left(\frac{2-q}{1-q}, \frac{1}{2}\right), & \text{for } q < 1 \end{cases} . \quad (2) \end{aligned}$$

If $q = 1$, the q-normal distribution reduces to the conventional normal distribution or Gaussian distribution

$$p_1(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \quad (3)$$

The q-normal distribution gives the maximum value of the Tsallis entropy which is one of the generalized entropy and is also a non-extensive entropy. As changing only one parameter q, the q-normal distribution can realize the distribution from the uniform distribution ($q \to 3$) with non-compact support to the uniform distribution ($q \to -\infty$) with compact support which size is twice the variance continuously, through the Cauchy distribution, 't-distribution' and the conventional Gaussian distribution.

3. The q-Gabor wavelet

The Gabor wavelet is

$$\begin{aligned} G_a^b(t) &= \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}} e^{-\frac{(t-b)^2}{2a^2\sigma^2}} \\ &\quad \times \left(e^{i\frac{\omega_0}{a}(t-b)} - e^{-\frac{1}{2}\sigma^2\omega_0^2}\right). \quad (4) \end{aligned}$$

We define the q-Gabor wavelet in the same manner as the conventional Gabor wavelet. Then the mother wavelet (the analyzing wavelet) of the q-Gabor wavelet for $q < 1$ is defined as follow,

$$\begin{aligned} &G_q(t|\sigma, \omega_0) \\ &= \begin{cases} \frac{1}{\sqrt{\frac{3-q}{1-q}}B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)\sigma}\left(1 - \frac{1-q}{3-q}\frac{t^2}{\sigma^2}\right)^{\frac{1}{1-q}} \\ \times\left\{e^{i\omega_0 t} - \left(\frac{1}{2}\sqrt{-\frac{3-q}{1-q}\sigma^2\omega_0^2}\right)^{-\frac{3-q}{2(1-q)}} \right. \\ \left. \times\Gamma\left(\frac{5-3q}{2(1-q)}\right) I_{\frac{3-q}{2(1-q)}}\left(\sqrt{-\frac{3-q}{q-1}\sigma^2\omega_0^2}\right)\right\}, \\ \quad \text{for } -\sqrt{\frac{3-q}{1-q}}\sigma \le t \le \sqrt{\frac{3-q}{1-q}}\sigma \\ 0, \quad \text{otherwise} \end{cases} . \quad (5) \end{aligned}$$

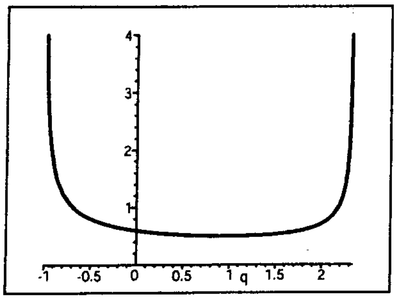


Figure 1. The uncertainty relation for the q-normal distribution. For $q \leq -1$, the variance Δ_ω^2 diverges. On the other hand, for $\frac{7}{3} \leq q$, the variance Δ_x^2 diverges. The minimum is attained at $q = 1$, that is, the conventional normal distribution gives the minimum uncertainty. But it is seen that the q-Gabor wavelet defined around $q = 1$ can be attained the value quite near the minimum.

where $I_v(x)$ is the modified Bessel function of the first kind and ω_0 is positive and called an analyzing frequency. The width of the time-frequency window Δ_x and the height of the time-frequency window Δ_ω for the q-Gabor wavelet are

$$\Delta_t = \sqrt{\frac{3-q}{7-3q}}\sigma\,, \qquad \text{for } q < \tfrac{7}{3} \tag{6}$$

$$\Delta_\omega = \sqrt{\frac{5-q}{2(3-q)(1+q)}}\frac{1}{\sigma}\,, \qquad \text{for } q > -1 \tag{7}$$

respectively. Therefore the uncertainty relation for the q-Gabor wavelet, which is called in the range of $-1 < q < \frac{7}{3}$, is as

$$\Delta_t\Delta_\omega = \sqrt{\frac{5-q}{2(7-3q)(1+q)}}\,, \qquad \text{for } -1 < q < \tfrac{7}{3}\,. \tag{8}$$

Similar to the conventional Gabor wavelet, the width and the height of the time-frequency window do not change in the length, depending on any spectrum of frequency. Figure 1 shows the uncertainty relation for the q-normal distribution. For $q \leq -1$, the variance Δ_ω^2 diverges. On the other hand, for $\frac{7}{3} \leq q$, the variance Δ_ω^2 diverges. The minimum is attained at $q = 1$, that is, the conventional normal distribution gives the minimum uncertainty. But it is seen that the q-Gabor wavelet defined around $q = 1$ can be attained the value quite near the minimum.

For $q < 1$, since the q-Gabor wavelet has the compact support, we can consider the discrete q-Gabor wavelet by replacing t with $2^b t - (m+1)\sigma\sqrt{\frac{3-q}{1-q}}$, where b and m are the scale (dilatation) and shift (translation) parameters respectively, and both b and m are integers. Then the discrete q-Gabor wavelet is

$$\begin{aligned} & G_q^{b,m}(t|\sigma,\omega_0) \\ &= \frac{\sqrt{2^b}}{\sqrt{\frac{3-q}{1-q}}B\left(\frac{2-q}{1-q},\frac{1}{2}\right)\sigma} \\ &\times\left\{1-\frac{1-q}{3-q}\frac{\left(2^b t-(2m+1)\sigma\sqrt{\frac{3-q}{1-q}}\right)^2}{\sigma^2}\right\}^{\frac{1}{1-q}} \\ &\times\left\{e^{-i\omega_0\left(2^b t-(2m+1)\sigma\sqrt{\frac{3-q}{1-q}}\right)}\right. \\ &\quad -\left(\frac{1}{2}\sqrt{-\frac{3-q}{1-q}\sigma^2\omega_0^2}\right)^{-\frac{3-q}{2(1-q)}}\Gamma\left(\frac{5-3q}{2(1-q)}\right) \\ &\quad \left.\times I_{\frac{3-q}{2(1-q)}}\left(i\omega\sigma\sqrt{\frac{3-q}{1-q}}\right)\right\} \end{aligned} \tag{9}$$

for $2^{-b}(2m\sigma\sqrt{\frac{3-q}{1-q}}) \leq t \leq 2^{-b}(2(m+1)\sigma\sqrt{\frac{3-q}{1-q}}))$. This wavelet is called the Type-1 q-Gabor wavelet. Figure 2 shows the Gabor wavelet and the Type-1 q-Gabor wavelet for various q with $b = 1$, $m = 0$, $\sigma = 1.0$, $\omega_0 = 1.0$.

On the other hand, another discrete q-Gabor wavelet can be constructed. Since the q-Gabor wavelet with $q < 1$ has the compact support with its width of $2\sqrt{\frac{3-q}{1-q}}\sigma$, when the analyzing frequency ω_0 is chosen such that the periodic time is proportional to the width of the support, that is,

$$\frac{2\pi n}{\omega_0} = 2\sqrt{\frac{3-q}{1-q}}\sigma\,, \tag{10}$$

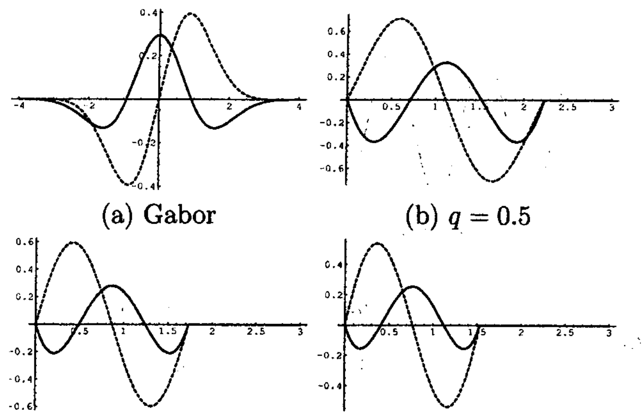
where n is an integer, then we have

$$\sigma = \frac{\pi n}{\sqrt{\frac{3-q}{1-q}}\omega_0} \qquad n = 1, 2, 3, \cdots. \tag{11}$$

In this case, we have the following discrete q-Gabor wavelet

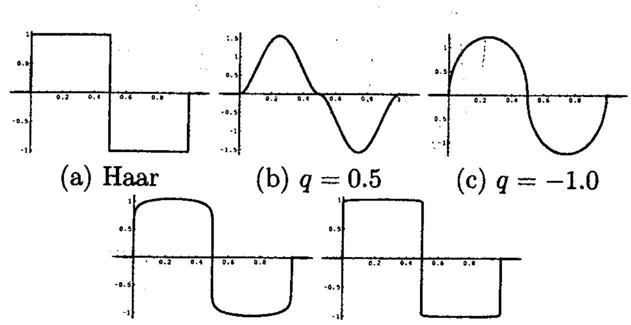
$$\begin{aligned} & G_q^m(t|n,\omega_0) \\ &= \sqrt{\frac{\omega_0}{n\pi}}\left\{\frac{\Gamma\left(\frac{7-3q}{2(1-q)}\right)^2}{\pi\Gamma\left(\frac{3-q}{1-q}\right)^2}\right\}^{\frac{1}{4}} \\ &\times\left\{1-\left(\frac{\omega_0}{n\pi}\right)^2\left(t-\frac{2mn\pi}{\omega}-\frac{\pi n}{\omega_0}\right)^2\right\}^{\frac{1}{1-q}} \\ &\times\left\{e^{i\omega\left(t-\frac{2mn\pi}{\omega}-\frac{n\pi}{\omega}\right)}-\left(\frac{2}{in\pi}\right)^{\frac{3-q}{2(1-q)}}\right. \\ &\quad \left.\times\Gamma\left(\frac{5-3q}{2(1-q)}\right)I_{\frac{3-q}{2(1-q)}}(in\pi)\right\}. \end{aligned} \tag{12}$$

This wavelet is called the Type-2 q-Gabor wavelet.



(a) Gabor (b) $q = 0.5$ (c) $q = 0.0$ (d) $q = -0.5$

Figure 2. The Gabor wavelet and the Type-1 q-Gabor wavelet for various q with $b = 1, m = 0, \sigma = 1.0, \omega_0 = 1.0$. The solid line stands for the real part and the dotted line for the imaginary part.



(a) Haar (b) $q = 0.5$ (c) $q = -1.0$ (d) $q = -10.0$ (e) $q = -100.0$

Figure 3. The Haar wavelet and the q-Haar wavelet for various q with $b = 0, m = 1$. For $q \to -\infty$, the q-Haar wavelet is equal to the Haar wavelet.

4. q-Haar wavelet

The Haar wavelet is

$$\psi(t) = \begin{cases} 1, & \text{for } 0 \le t < \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise} \end{cases} \quad (13)$$

If two sets of the q-normal distributions are connected anti-symmetrically, we can get the q-Haar wavelet as a possible generalization of the Haar wavelet. Then the q-Haar wavelet is

$$\psi_q^{b,m}(t) = \begin{cases} \sqrt{2^{b-1}} \left(\frac{\sqrt{\pi}}{4} \frac{\Gamma(\frac{3-q}{1-q})}{\Gamma(\frac{7-3q}{2(1-q)})} \right)^{-\frac{1}{2}} \\ \times \left(1 - \left(2^{b+2}t - (4m+1)\right)^2\right)^{\frac{1}{1-q}}, \\ \quad \text{for } 2^{-b} \le t \le 2^{-b}(m + \frac{1}{2}) \\ -\sqrt{2^{b-1}} \left(\frac{\sqrt{\pi}}{4} \frac{\Gamma(\frac{3-q}{1-q})}{\Gamma(\frac{7-3q}{2(1-q)})} \right)^{-\frac{1}{2}} \\ \times \left(1 - \left(2^{b+2}t - (4m+3)\right)^2\right)^{\frac{1}{1-q}}, \\ \quad \text{for } 2^{-b}(m + \frac{1}{2}) \le t \le 2^{-b}(m+1) \\ 0, \quad \text{otherwise} \end{cases} . \quad (14)$$

where b and m are integers. Figure 3 shows the Haar wavelet and the q-Haar wavelet for various q with $b = 0$, $m = 1$. For $q \to -\infty$, the q-Haar wavelet is equal to the Haar wavelet.

5. Experiments

We give experiments on the Gabor wavlet, the Haar wavelet, the Type-1 q-Gabor wavelet, the Type-2 q-Gabor wavelet and the q-Haar wavelet. Figure 4 shows examples of the Gabor wavelet transform and the Haar wavelet transform. (a) is input signal for the Gabor wavelet transform, $y(t) = \cos 4t$. (b) is Gabor wavelet transform of (a). (c) is input signal for the Haar wavelet transform, $y(t) = \cos \frac{\pi}{2} t$. (d) is Haar wavelet transform of (c). Figure 5 and Figure 6 show examples of the Type-1 q-Gabor wavelet, the Type-2 q-Gabor wavelet and the q-Haar wavelet for various parameter q. Input signal is given as Figure 4(a). Figure 7 shows examples of the Haar wavelet for various parameter q. Input signal is given as Figure 4(a). The vertical axes represents scale (dilation) and the horizontal axes shows shift (translation).

Figure 5(e) is similar to Figure 4(b). The Type-1 q-Gabor wavelet with $q = -0.3$ has most effect for the input signal, $y(t) = \cos 4t$. Therefore, we can estimate the width of input signal using the information, $q = -0.3$.

In Figure 7, all examples are similar to Figure 4(d). It is found that the q-Haar wavelet has similar property to Haar wavelet.

Further research is to make system which decide paramter q automatically for various input signal.

Acknowledgments

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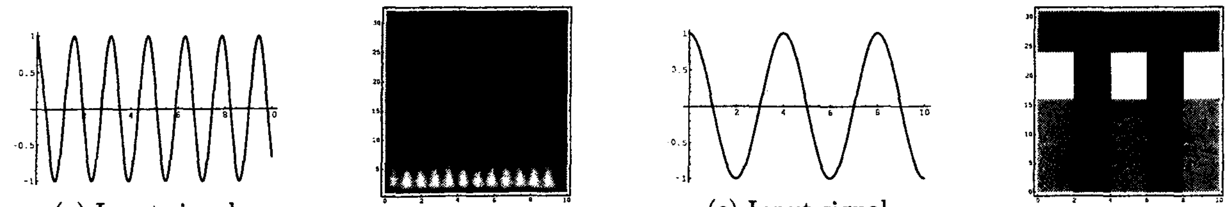
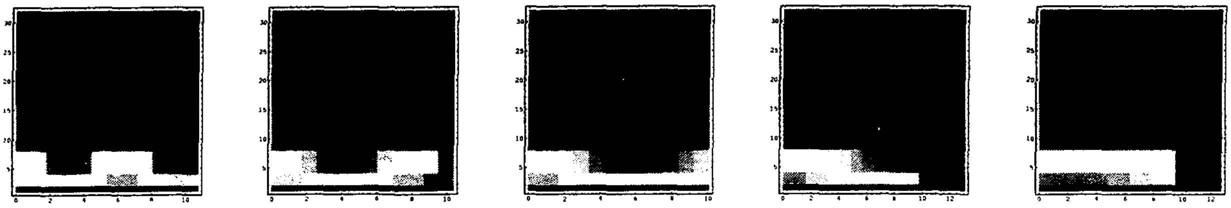


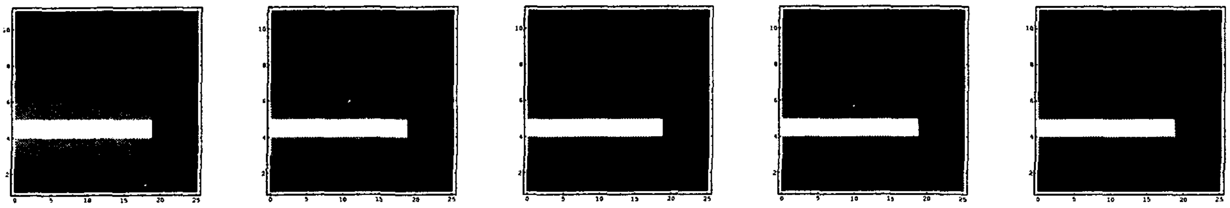
(a) Input signal (b) Gabor wavelet (c) Input signal (d) Haar wavelet

Figure 4. Example of the Gabor and Haar wavelet transform. (a) is input signal for Gabor wavelet transform, $y(t) = \cos 4t$. (b) is Gabor wavelet transform of (a). (c) is input signal for Haar wavelet transform, $y(t) = \frac{\pi}{2}t$. (d) is Haar wavelet transform of (c). The vertical axes of (b) represents scale and the horizontal axes shows shift.



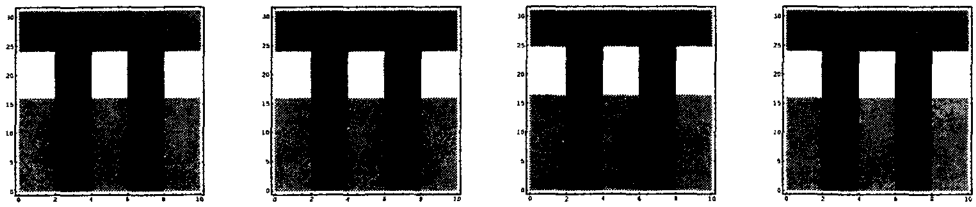
(a) $q = 0.1$ (b) $q = 0.0$ (c) $q = -0.1$ (d) $q = -0.2$ (e) $q = -0.3$

Figure 5. Examples of the Type-1 q-Gabor wavelet transform for various q with $\sigma = 1.0, \omega = 1.0$. Input signal is given as Figure 4(a). The vertical axes represents scale (dilation) and the horizontal axes shows shift (translation).



(a) $q = 0.7$ (b) $q = 0.6$ (c) $q = 0.5$ (d) $q = 0.4$ (e) $q = 0.3$

Figure 6. Examples of the Type-2 q-Gabor wavelet transform for various q with $n = 1$. Input signal is given as Figure 4(a). The vertical axes represents scale (dilation) and the horizontal axes shows shift (translation).



(a) $q = 0.5$ (b) $q = -1.0$ (c) $q = -10.0$ (d) $q = -100.0$

Figure 7. Examples of the q-Haar wavelet transform for various q. Input signal is given as Figure 4(c). The vertical axes represents scale (dilation) and the horizontal axes shows shift (translation).