The properties of the two dimensional q-Gabor wavelet

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Abstract: In this paper, we give the definition of the two dimensional q-Gabor wavelet. It consists of the q-normal distribution, which is also given in this paper. If the q-normal distribution is used as a kernel of the Gabor wavelet instead of the normal distribution, the q-Gabor wavelet is obtained. Furthermore, the q-Gabor wavelet is compared with the Gabor and the Haar wavelets to show how good the q-Gabor wavelet is.

1. Introduction

In this paper, we give the definition of the two dimensional q-Gabor wavelet. It consists of the q-normal distribution, which is also given in this paper. If the q-normal distribution is used as a kernel of the Gabor wavelet instead of the normal distribution, the q-Gabor wavelet is obtained.

On the uncertainty relation, the Gabor wavelet, which is the Gaussian modulated sinusoidal function, has the minimum time-frequency window. But the width and the height of the time window and the frequency window are constant about the analysis frequency. On the Haar wavelet, which is the only actual orthonormal wavelet, the width and the height of the time window are variable. But the time-frequency window size is infinity. On the other hand, the q-Gabor wavelet has the time-frequency window as small as the Gabor wavelet, and the width and the height are variable.

We show the equation of the q-normal distribution, and define the q-Gabor wavelet based on the q-normal distribution. In addition, we extend the q-Gabor wavelet to two dimensions, and analyze the digital image by the two dimensional q-Gabor wavelet. Compared with the known wavelets, this wavelet has possibility that be able to show more good result on two dimensional analysis.

Below,we give the brief review of the q-normal distribution and the q-Gabor wavelet, and show the experiments of IWT(integral wavelet transform) and DWT(discrete wavelet transform) for digital image by the two dimensional q-Gabor wavelet.

2. q-normal distribution

The q-normal distribution, which is the probability density function, smoothly vary along with parameter q. The q-normal distribution stands for the Gaussian distribution for the case of $q = 1$. And also stands for the Haar scaling function for the case of $q = -\infty$. Especially if $q < 1$, the q-normal distribution has the compact support. The q-normal distribution is shown following equation.

$$P_q(t) = \frac{1}{\sigma\sqrt{\frac{3-q}{1-q}}} \frac{\Gamma\left(\frac{5-3q}{2(1-q)}\right)}{\Gamma\left(\frac{2-q}{1-q}\right)} \left\{1 - \frac{1-q}{3-q}\frac{x^2}{\sigma^2}\right\}^{\frac{1}{1-q}} \quad (1)$$

The width of the support is given by $2\sigma\sqrt{\frac{3-q}{1-q}}$. The reason of this is the argument $1 - \frac{1-q}{3-q}\frac{x^2}{\sigma^2}$ of above equation should be positive number. The q-normal distribution takes zero in the outside of the support. If this function is used as a kernel of the Gabor wavelet instead of the normal distribution, we can obtain the q-Gabor wavelet. Then the q-Gabor wavelet is defined as the "orthonormal" wavelet. Here "orthonormal" means that the q-Gabor wavelet is orthogonal about the shift and normalized 1 for the squared absolute value of the q-Gabor wavelet.

On the uncertainty relation, the normal distribution has the minimum time-frequency window size. The width and the height of the time-frequency window are given by following equations.

$$\triangle_t = \frac{1}{\sqrt{2}}\sigma \quad (2)$$

$$\triangle_\omega = \frac{1}{\sqrt{2}}\frac{1}{\sigma} \quad (3)$$

From above equations, the size of time-frequency window is $\triangle_t \triangle_\omega = \frac{1}{2}$.

On the other hand, the these of the q-Gabor wavelet are given by following equations.

$$\triangle_t = \sigma\sqrt{\frac{3-q}{7-3q}}, \quad \text{for} \quad q < \frac{7}{3} \quad (4)$$

$$\triangle_\omega = \frac{1}{\sigma}\sqrt{\frac{5-q}{2(3-q)(1+q)}}, \quad \text{for} \quad q > -1 \quad (5)$$

The Figure1 shows the relation between the time-frequency window size and the parameter q. If $q \leq -1$ or $\frac{7}{3} \leq q$, the window size diverges. On $q = 1$, the minimum time-frequency window is obtained.

3. q-Gabor wavelet

The q-Gabor wavelet consists of the q-normal distribution, therefore the q-Gabor wavelet carries out different behavior in $q < 1$ and $1 \leq q$. If $q < 1$, there are two

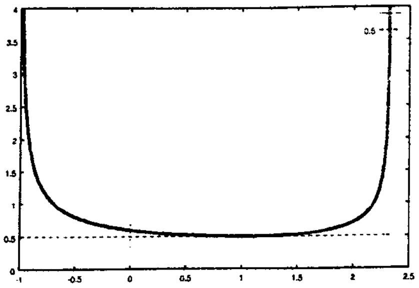


Figure 1. The time-frequency window size about variance of q

types of the q-Gabor wavelet. One has σ which is continuous about variance(type1), the other, σ is discretized by n(type2). Here n is the natural number. Both of them, behave as the discrete wavelet, and has the compact support. We define the type1 q-Gabor wavelet as the equation (6).

$$g_q^m(x) = P_q(x)\left\{e^{-i\omega x} - I_{\frac{3-q}{2(1-q)}}\left(-i\omega\sigma\sqrt{\frac{3-q}{1-q}}\right)\right.$$
$$\left.\times\Gamma\left(\frac{5-3q}{2(1-q)}\right)2^{\frac{3-q}{2(1-q)}}\left(-\sigma^2\omega^2\frac{3-q}{1-q}\right)^{-\frac{3-q}{4(1-q)}}\right\} \quad (6)$$

Here, I_ν is the modified Bessel function of the first kind. And P_q stands for the normalized q-normal distribution. It is defined as follows.

$$P_q(t) = \sqrt{2^b}\left\{\sigma\sqrt{\frac{3-q}{1-q}}\pi\frac{\Gamma\left(\frac{3-q}{1-q}\right)}{\Gamma\left(\frac{7-3q}{2(1-q)}\right)}\right\}^{-\frac{1}{2}}$$
$$\times\left(1-\frac{1-q}{3-q}\frac{x^2}{\sigma^2}\right)^{\frac{1}{1-q}} \quad (7)$$

The width of type1 q-Gabor wavelet is given by $2\sigma\sqrt{\frac{3-q}{1-q}}$. Then we assume that the periodic time is proportional to the width of the support. The equation (8) shows this relation.

$$2\frac{n\pi}{\omega} = 2\sigma\sqrt{\frac{3-q}{1-q}} \quad (8)$$

From this relation, we obtain the q-Gabor wavelet which σ is discretized. So we define the type2 q-Gabor wavelet as the equation (9).

$$g_q(t) = P_q\left[e^{-i\omega x} - I_{\frac{3-q}{2(1-q)}}(in\pi)\Gamma\left(\frac{5-3q}{2(1-q)}\right)\right.$$
$$\left.\times 2^{\frac{3-q}{2(1-q)}}(in\pi)^{-\frac{3-q}{2(1-q)}}\right] \quad (9)$$

On the type2 q-Gabor wavelet, P_q is defined as follows.

$$P_q(t) = \sqrt{\frac{\omega}{n\pi}}\left\{\sqrt{\pi}\frac{\Gamma\left(\frac{3-q}{1-q}\right)}{\Gamma\left(\frac{7-3q}{2(1-q)}\right)}\right\}^{-\frac{1}{2}}$$
$$\times\left[1-\left(\frac{\omega}{n\pi}\right)^2x^2\right]^{\frac{1}{1-q}} \quad (10)$$

On the type2 q-Gabor wavelet, the time and the frequency window width are given by following equations.

$$\triangle_x = \sqrt{\frac{3-q}{7-3q}\frac{n\pi}{\omega}} \qquad q < \frac{7}{3} \quad (11)$$
$$\triangle_\omega = \sqrt{\frac{5-q}{2(1+q)(3-q)}\frac{\omega}{n\pi}} \qquad -1 < q < 3 \quad (12)$$

Therefore, the time-frequency window size can be obtained by following equation.

$$\triangle_x\triangle_\omega = \sqrt{\frac{5-q}{2(7-3q)(1+q)}} \quad (13)$$

On the above equation, if $q = 1$, the window size becomes $\frac{1}{2}$. It is same window size as that of the Gabor wavelet, and it is the minimum window size in all wavelets. Though, on the Gabor wavelet, from equations (2) and (3), the width and the height of the time-frequency window are constant about the analyzing frequency ω. But on the type2 q-Gabor wavelet,from equations (7) and (8), it is figured out that the time and the frequency window width vary along with ω. This feature is more advantageous than the Gabor wavelet for analysis on very high or low frequency.

Below, we extend the one dimensional q-Gabor wavelet to two dimensions.

On two dimensions, it can be considered that three types of the q-Gabor wavelet. The three types are the type1 each case of $q < 1$, $1 \le q$, and the type2 on $q < 1$. On every case, the equation of wavelet is not different from that of the one dimensional q-Gabor wavelet, but the equation of the q-normal distribution is. The following equation shows the normalized type2 q-normal distribution on two dimensions.

$$P_q(x,y) = \frac{\omega}{\pi}\left(n_x n_y \pi\frac{1-q}{3-q}\right)^{-\frac{1}{2}}$$
$$\times\left\{1-\left(\frac{\omega}{n_x\pi}\right)^2x^2+\left(\frac{\omega}{n_y\pi}\right)^2y^2\right\}^{\frac{1}{1-q}} \quad (14)$$

We must be careful about the upper bound of the q. From the condition, that of the argument of the Gamma function must be positive number, $q = 3$ is upper bound of q on the one dimensional q-Gabor wavelet. On two dimensions, it is $q = 2$.

4. Experiments

By the Gabor, the type1 and the type2 q-Gabor wavelet, we do WT(wavelet transform) for the digital image which presented by Figure 2. The details of the experiment is follows. On the Gabor wavelet, we do IWT. On the type2 q-Gabor and the Haar wavelet, we do DWT. The type1 q-Gabor wavelet can do both wavelet transforms. The Gabor, and the q-Gabor wavelet can be rotated, so it can analyze the image on any direction. The original image is analyzed

Figure 2. Original image

for $\theta = 0°, 45°$ and $90°$ directions in this experiment. As to the Haar wavelet, first, we do WT for horizontal direction. Next, do it for vertical direction. Then we obtain the analyzed images for horizontal($\theta = 0°$), vertical($\theta = 90°$), and diagonal($\theta = 45°$) direction.

From the Figure 3 to 7 are the results of WT by the each wavelets, and they show the edges of the original image for each directions. Compared Figure3 with 4, they show almost the same results. But the case of the q-Gabor wavelet, we can see detail information. For example, heir of the girl on $\theta = 45°$ is more wavy than that of the Gabor wavelet.

Figure 5 is the DWT by the Haar wavelet. Figure 6 and 7 are that by the type1 and type2 q-Gabor wavelet on the case of $q = 0.5$. On this experiment, we determined the parameter to output same resolution images. On MRA(Multi Resolutional Analysis), this resolution is called "level". The case of the type2 q-Gabor wavelet, we can determine the accuracy of resolution by $\frac{2n\pi}{\omega}$, and it is 4 on figure 7. About the DWT, on each direction, the q-Gabor wavelet shows the more detail results than the Haar wavelet. Especially the type2 q-Gabor wavelet can vary its time-frequency window width by n. So the DWT by the type2 q-Gabor wavelet is very easy, but the results has high accuracy.

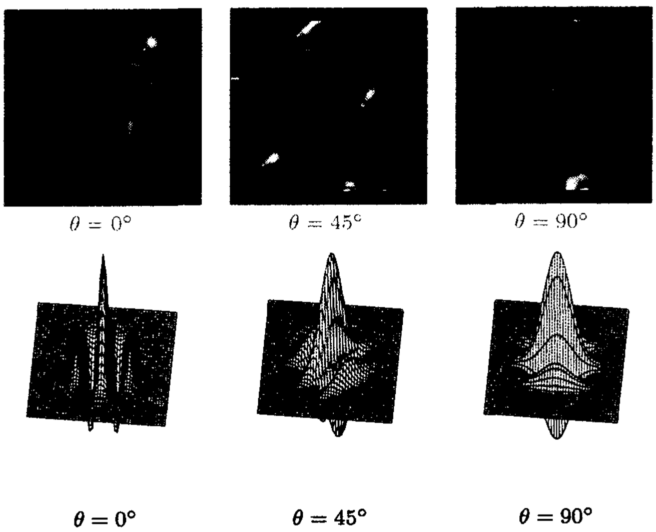


Figure 3. IWT by the Gabor wavlet with $\omega = \frac{\pi}{4}$ and $\sigma = 4$

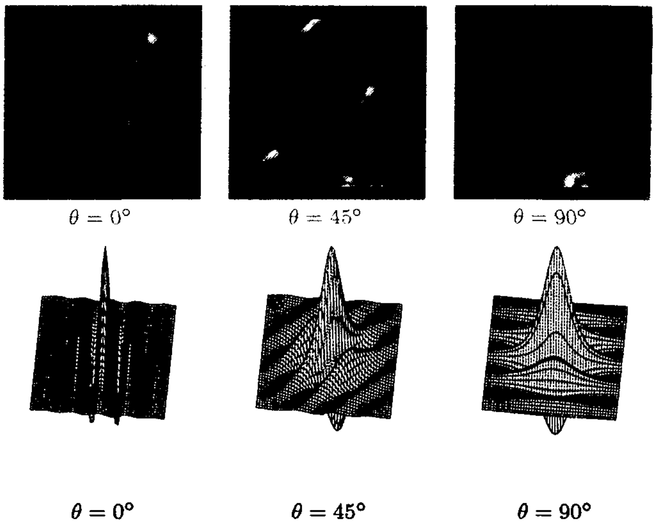


Figure 4. IWT by the type1 q-Gabor wavlet with $q = 1.5, \omega = \frac{\pi}{4}$ and $\sigma = 4$

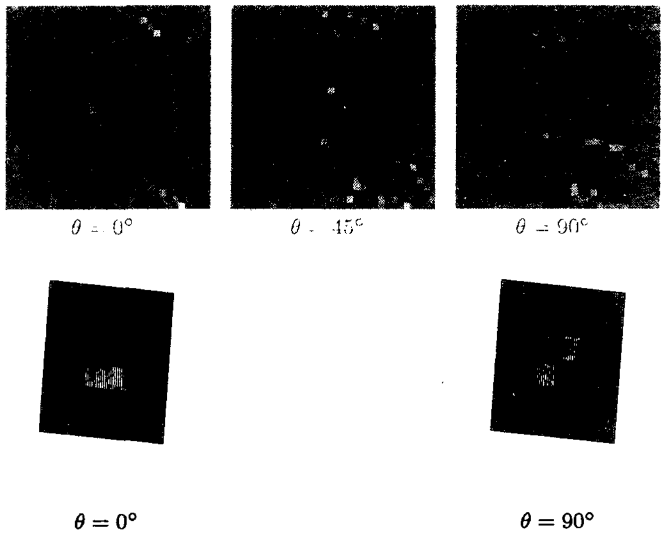


Figure 5. DWT by the Haar wavelet

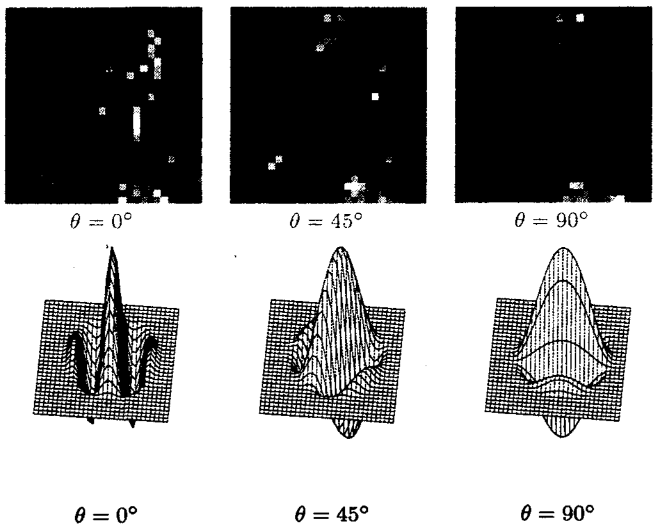
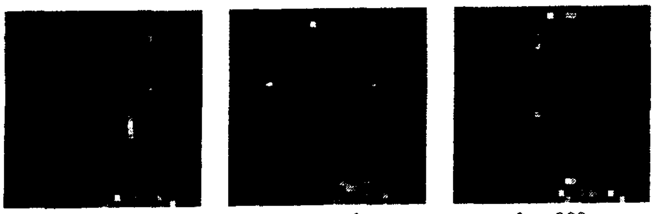
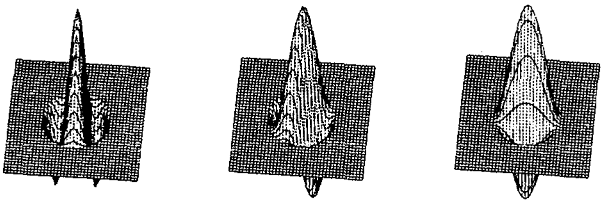


Figure 6. DWT by the type1 q-Gabor wavlet with $q = 0.5, \omega = \frac{\pi}{2}$ and $\sigma = 4$



$\theta = 0°$ $\theta = 45°$ $\theta = 90°$



$\theta = 0°$ $\theta = 45°$ $\theta = 90°$

Figure 7. DWT by the type2 q-Gabor wavelet with $q = 0.5$, $\omega = \frac{\pi}{2}$ and $n = 2$

5. Conclusion

From the one dimensional q-normal distribution, we defined the two dimensional q-normal distribution. And we also difined the two dimensional q-Gabor wavelet. It shows different behavior on $q < 1$, and $1 \leq q$. Furthermore, on $q < 1$, it has compact support, and there are two types of the q-Gabor wavelet, called type1, and type2. Especially the type2 q-Gabor wavelet has time-frequency window which vary along with analyzing frequency ω. This feature gaves us advatage for analysis. But we must be careful about the upper bound of the q, on two dimension. For one dimension it is $q = 3$, and $q = 2$ on two dimension.

From the results of experiments, for the q-Gabor wavelet has equivalent analysis capability to the Gabor wavelet on $1 \leq q$. On $q < 1$ it shows superior analysis capability to the Haar wavelet. And its time-frequency window size can be varied by n for type2 case.

Thus, while the q-Gabor wavelet is the single wavelet, it behaves as the continuous wavelet on $1 \leq q$, and behaves as the discrete one on $q < 1$. Furthermore its performance is more better than or equal to the known wavelets. This extends the application area of the q-Gabor wavelet.

References

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