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# 정식화를 이용한 3차원 구조물의 형상 최적설계

## Variational Formulation for Shape Optimization of Spatial Beam Structures

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## ABSTRACT

A general formulation for shape design sensitivity analysis over three dimensional beam structure is developed based on a variational formulation of the beam in linear elasticity. Sensitivity formula is derived based on variational equations in cartesian coordinates using the material derivative concept and adjoint variable method for the displacement and Von-Mises stress functionals. Shape variation is considered for the beam shape in general 3-dimensional direction as well as for the orientation angle of the beam cross section. In the sensitivity expression, the end points evaluation at each beam segment is added to the integral formula, which are summed over the entire structure. The sensitivity formula can be evaluated with generality and ease even by employing piecewise linear design velocity field despite the bending model is fourth order differential equation. For the numerical implementation, commercial software ANSYS is used as analysis tool for the primal and adjoint analysis. Once the design variable set is defined using ANSYS language, shape and orientation variation vector at each node is generated by making finite difference to the shape with respect to each design parameter, and is used for the computation of sensitivity formula. Several numerical examples are taken to show the advantage of the method, in which the accuracy of the sensitivity is evaluated. The results are found excellent even by employing a simple linear function for the design velocity evaluation. Shape optimization is carried out for the geometric design of an archgrid and tilted bridge, which is to minimize maximum stress over the structure while maintaining constant weight. In conclusion, the proposed formulation is a useful and easy tool in finding optimum shape in a variety of the spatial frame structures.

Keywords : Shape optimization, Variational formulation, Design sensitivity analysis, Beam structure

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## 1. Introduction

In the design of various structural systems, many of the components include beams and its combination, which can be found for example in arch bridges, aircraft wing and vehicle frames. The objective of the present paper is to formulate and implement design sensitivity analysis and apply it to the shape optimization of complex beam structures. Shape sensitivity analysis has long been a subject in various engineering fields, of which there have been two major directions. One is finite dimensional approach where the sensitivity is derived and evaluated from the discretized formulations [1]. The other is continuum or variational approach, which derives analytic sensitivity and evaluate numerically afterwards [2]. Present paper belongs to the latter approach. Though there are plenty of literatures regarding the beam shape optimizations, they are mostly focused at finding optimality criteria or algorithm on ad-hoc basis [3-5]. Not many literatures are found for the general approach utilizing the sensitivity information of the frame structures. Dems and Mroz [6] studied plane arch shape design sensitivity, in which a general procedure is presented using variational calculus but is limited to the 2 dimensional structures. Twu and Choi [7,8] studied configuration design sensitivity in which unified formula is presented for the general 3 dimensional structures. In their formulation for the beam structure, shape variation of the beam was decomposed into translation and rotation and sensitivity formula was derived separately for each variation. However the sensitivity computation was not straightforward and the accuracy was not satisfactory when the design velocity function was linear, which was solved by either calculating corner terms or employing cubic function for the velocity. Later on, Park and Choi [9] presented another approach in applying the formulation to nonlinear system, in which Timoshenko beam model was employed to reduce the derivative order of the design velocity field.

In this paper, general formulation is developed using the material derivative concept and adjoint variable method for the three-dimensional beam structure. The formulation considers not only the shape variation which includes translation as well as rotation of the beam axis but also the orientation variation of the beam cross section. Once the variational equation is derived from the original Euler beam theory, subsequent procedure including the velocity field is described based on the global Cartesian coordinates, of which the axes are fixed with respect to the configuration change. Computation of the obtained sensitivity formula is quite straightforward and easy to implement even by employing piecewise linear design velocity field. Commercial code ANSYS is employed for the structural analysis, and design sensitivity analysis is implemented using the post-processing data. Several numerical examples of sensitivity analysis are taken to show the advantage of the method. Shape optimization is also conducted to illustrate the excellent applicability.

## 2. Shape Design Sensitivity Analysis

Consider a beam segment $\Gamma_s$ in an arbitrary spatial frame structure as shown in Fig.1. The beam shape is described by a local coordinate $(s,t,n)$ as in Figure 2(a) where the neutral axis is given by the point vector $\mathbf{x}=\{x,y,z\}$ where $x=x(s), y=y(s), z=z(s)$. The two normal directions $(t,n)$ can be chosen for convenience at the principal axes of moments of inertia. Due to a distributed load vector $\mathbf{p}=\{p_i\}$, which includes concentrated loads in a Dirac measure, the beam segment undergoes axial and shear loads as well as moments along the beam, which are denoted by the vectors $\mathbf{P}=\{P,Q,R\}$ and $\mathbf{M}=\{M_s,M_t,M_n\}$, respectively, as shown in Figure 2(a). The deformations of the beam due to these loads are expressed by displacement $\mathbf{u}=\{u,v,w\}$ and rotation $\boldsymbol{\varphi}=\{\varphi_s,\varphi_t,\varphi_n\}$ as shown in Figure 2(b). From the weighted residual formulations of force and moment equilibrium equations, one can derive a variational identity for the beam segment in the following form

$$a_s(\mathbf{u},\bar{\mathbf{u}})=l_s(\mathbf{u},\bar{\mathbf{u}}) \tag{1}$$

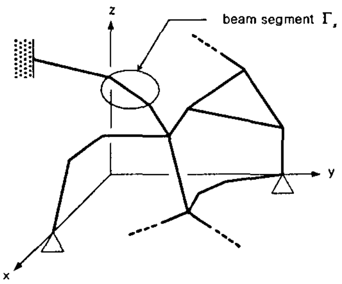
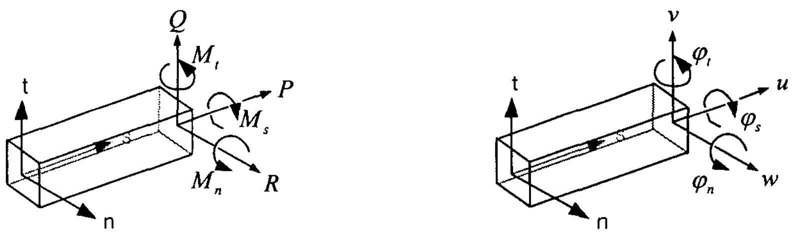



Figure 1. Three-dimensional beam structure.




(a) (b)

Figure 2. Beam segment under loading in local coordinate.
(a) Forces and moments; (b) displacements and rotations.

where $a_s$ and $l_s$ are the energy bilinear and load linear functional respectively, given by

$$\begin{aligned} a_s(\mathbf{u},\bar{\mathbf{u}}) &= \int_{\Gamma_s}\{C\varepsilon(\mathbf{u})\varepsilon(\bar{\mathbf{u}})+D_n\kappa_n(\mathbf{u})\kappa_n(\bar{\mathbf{u}})+D_t\kappa_t(\mathbf{u})\kappa_t(\bar{\mathbf{u}})+B\gamma(\mathbf{u})\gamma(\bar{\mathbf{u}})\}ds \\ l_s(\mathbf{u},\bar{\mathbf{u}}) &= \int_{\Gamma_s}\mathbf{p}\cdot\bar{\mathbf{u}}ds+[\mathbf{P}\cdot\bar{\mathbf{u}}+\mathbf{M}\cdot\mathrm{j}(\bar{\mathbf{u}})]_{\Gamma_s} \end{aligned} \tag{2}$$

in which the subscript $s$ denotes the beam segment and the over-bar symbol denotes the arbitrary weight variable. The vector $\mathbf{u}$ includes $\{u,v,w,\varphi_s\}$ since the true unknowns are $u,v,w$ and $\varphi_s$. The squared bracket $[\cdot]_{\Gamma_s}$ means the difference between the both end points of the segment. The strains in this equation are respectively given as

$$\varepsilon = u_{,s},\ \varphi_t = -w_{,s},\ \varphi_n = v_{,s}$$
$$\gamma = \varphi_{s,s},\ \kappa_t = \varphi_{t,s},\ \kappa_n = \varphi_{n,s}$$

and the constants are given by

$$\begin{aligned} &C = EA,\quad D_t = EI_t,\quad D_n = EI_n, \\ &B = GJ = G(I_t + I_n) \end{aligned} \tag{3}$$

where $E,G$ are elastic and shear modulus, $A,I_t,I_n,J$ are area and moment of inertia with respect to $t,n,s$ axis respectively.
The formulation can also be expressed in terms of Cartesian components where the displacements are $\mathbf{z} = \{z_i;\varphi_s\}$, rotations are $\mathrm{f} = \{\phi_i\}$ and the forces and moments are $\mathbf{f} = \{f_i\}$, $\mathbf{F} = \{F_i\}$ and $\mathbf{N} = \{N_i\}$. Then the identity becomes

$$a_s(\mathbf{z}, \overline{\mathbf{z}}) = l_s(\mathbf{z}, \overline{\mathbf{z}}) \tag{4}$$

where

$$\begin{aligned} &a_s(\mathbf{z},\overline{\mathbf{z}}) = \int_{\Gamma_s} \{C\varepsilon(\mathbf{z})\varepsilon(\overline{\mathbf{z}}) + D_n\kappa_n(\mathbf{z})\kappa_n(\overline{\mathbf{z}}) + D_t\kappa_t(\mathbf{z})\kappa_t(\overline{\mathbf{z}}) + B\gamma(\mathbf{z})\gamma(\overline{\mathbf{z}})\}ds \\ &l_s(\mathbf{z},\overline{\mathbf{z}}) = \int_{\Gamma_s} \mathbf{f}\cdot\overline{\mathbf{z}}ds + [\mathbf{F}\cdot\overline{\mathbf{z}} + \mathbf{N}\cdot\mathrm{f}(\overline{\mathbf{z}})]_{\Gamma_s} \end{aligned} \tag{5}$$

Assemble the identity into the entire structure, and notice that

$$\sum_s [\mathbf{F}\cdot\overline{\mathbf{z}} + \mathbf{N}\cdot\overline{\mathrm{f}}]_{\Gamma_s} = \sum_j \{(\Sigma\mathbf{F})\cdot\overline{\mathbf{z}} + (\Sigma\mathbf{N})\cdot\overline{\mathrm{f}}\} = \sum_j (\mathbf{F}^p\cdot\overline{\mathbf{z}} + \mathbf{N}^p\cdot\overline{\mathrm{f}}) \tag{6}$$

Applying suitable boundary conditions for the weighted displacement $\overline{\mathbf{z}}$, one gets the variational equation for the global structure

$$a(\mathbf{z}, \overline{\mathbf{z}}) = l(\overline{\mathbf{z}}) \tag{7}$$

where $a$ is simply the summation of integrals in $a_s$, and

$$l(\overline{\mathbf{z}}) = \int_\Gamma \mathbf{f}^p\cdot\overline{\mathbf{z}}ds + \sum_s [\mathbf{F}\cdot\overline{\mathbf{z}} + \mathbf{N}\cdot\overline{\mathrm{f}}]_{\Gamma_s} = \int_\Gamma \mathbf{f}^p\cdot\overline{\mathbf{z}}ds + \sum_j (\mathbf{F}^p\cdot\overline{\mathbf{z}} + \mathbf{N}^p\cdot\mathrm{f}(\overline{\mathbf{z}})) \tag{8}$$

Note from here that $\mathbf{f}$ is replaced by $\mathbf{f}^p$, in which the superscript $p$ has the meaning of the prescribed load for primal system.

A displacement functional which includes displacement $\mathbf{z}$ and rotation $\mathrm{f}$ in cartesian coordinate, defined at a point $\mathbf{x}_0$ is considered in the form

$$\Phi_d = \psi(\mathbf{z}_0, \mathrm{f}_0) = \int_\Gamma \psi(\mathbf{z}, \mathrm{f})\delta(\mathbf{x}, \mathbf{x}_0)ds \tag{9}$$

Material derivative concept is employed to express the shape variation, which is defined as a transformation process by velocity vector $\mathbf{V}(\mathbf{x})$ for translation and by angular velocity $\omega(\mathbf{x})$ for orientation angle. The sensitivity formula is then derived by taking derivative and employing adjoint variable. The resulting formula is expressed as

$$\Phi_d' = \int_\Gamma \mathbf{f} \cdot \mathbf{z}^* V_{s,s} ds + l'\left(\mathbf{z}, \mathbf{z}^*; \mathbf{V}\right) - a'(\mathbf{z}, \mathbf{z}^*; \mathbf{V}) \tag{10}$$

where

$$a'(\mathbf{z}, \mathbf{z}^*; \mathbf{V}) = \int_\Gamma \begin{bmatrix} C\left(\varepsilon_1 \varepsilon^* + \varepsilon \varepsilon_1^*\right) + D_t\left(\kappa_{t1} \kappa_t^* + \kappa_t \kappa_{t1}^*\right) + D_n\left(\kappa_{n1} \kappa_n^* + \kappa_n \kappa_{n1}^*\right) + \\ B\left(\gamma_1 \gamma^* + \gamma \gamma_1^*\right) + (C \varepsilon \varepsilon^* + D_n \kappa_n \kappa_n^* + D_t \kappa_t \kappa_t^* + B \gamma \gamma^*) V_{s,s} \end{bmatrix} ds$$

$$l'\left(\mathbf{z}, \mathbf{z}^*; \mathbf{V}\right) = \sum_s \left[\mathbf{M} \cdot \mathrm{j}_2(\mathbf{z}^*; \mathbf{V}) + \mathbf{M}^* \cdot \mathrm{j}_2\left(\mathbf{z}; \mathbf{V}\right)\right]_{\Gamma_s} \tag{11}$$

where variables subscripted with 1 and 2 denote derived variables as a result of material derivative, which include design velocity $\mathbf{V}$ and $\boldsymbol{\omega}$. They are not listed here for brevity. Adjoint solution $\mathbf{z}^*$ is obtained by imposing the load and moment $\mathbf{F}^a, \mathbf{N}^a$ which are obtained from

$$\sum_j \left(\mathbf{F}^a \cdot \dot{\mathbf{z}} + \mathbf{N}^a \cdot \dot{\mathrm{f}}\right) = \left(\psi_{,\mathbf{z}} \cdot \dot{\mathbf{z}} + \psi_{,\mathrm{f}} \cdot \dot{\mathrm{f}}\right)\Big|_{\mathbf{x}=\mathbf{x}_0} \tag{12}$$

Consider next a stress functional which represents an average value over a small segment of the beam $\Gamma_\varepsilon$

$$\Phi_s = \int_{\Gamma_\varepsilon} \sigma_e ds \Big/ \int_{\Gamma_\varepsilon} ds = \int_{\Gamma_\varepsilon} \sigma_e ds / l_\varepsilon \tag{13}$$

Then the sensitivity is obtained in the form

$$\Phi_s' = \int_\Gamma \mathbf{f} \cdot \mathbf{z}^* V_{s,s} ds + l'\left(\mathbf{z}, \mathbf{z}^*; \mathbf{V}\right) - a'(\mathbf{z}, \mathbf{z}^*; \mathbf{V}) + R(\mathbf{z}; \mathbf{V}) \tag{14}$$

where $R$ is just a collection of terms that are explicit in terms of velocity vector,

$$R(\mathbf{V}) = \int_{\Gamma_\varepsilon} \{\sigma_{e,\sigma} \sigma_1\left(\mathbf{z}; \mathbf{V}\right) + \sigma_{e,\tau} \tau_1\left(\mathbf{z}; \mathbf{V}\right)\} ds + \int_{\Gamma_\varepsilon} (\sigma - \Phi) V_{s,s} ds \tag{15}$$

As in the previous equation, variables subscripted with 1 and 2 denote derived variables as a result of material derivative. Adjoint loads are defined such that

$$\begin{aligned} &\int_\Gamma \mathbf{f}^a \cdot \dot{\mathbf{z}} ds + \sum_j \left(\mathbf{F}^a \cdot \dot{\mathbf{z}} + \mathbf{N}^a \cdot \dot{\mathrm{f}}\right) = \\ &\int_{\Gamma_\varepsilon} \left[\sigma_{e,\sigma} E\left\{\varepsilon(\dot{\mathbf{z}}) + \mu_t \dot{\mathrm{f}}_{,s} \cdot \mathbf{t} + \mu_n \dot{\mathrm{f}}_{,s} \cdot \mathbf{n}\right\} + \sigma_{e,\tau} G \dot{\mathrm{f}}_{,s} \cdot \mathbf{s}\right] ds \end{aligned} \tag{16}$$

## 3. Example of Shape Design Sensitivity Analysis

Example 1: Three-bar frame

Consider a 3-beam frame as shown in Fig.3, which was studied by Twu and Choi [8]. Displacement at the point of load and 3 stress functionals at each segment are considered.

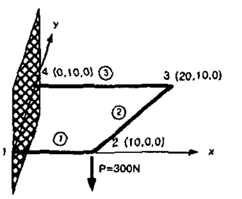


Figure 3. 3-beam frame problem

Sensitivity predictions are compared with those by finite differences by 0.1% design perturbation Results with design variable x of node 3 as well as orientation angle of segment 3 are given in Table 1 and 2.

Table 1 Sensitivity w.r.t. x coord. of node 3

| | Values | Sensitivity by FD | Sensitivity by DSA | Ratio(%) |
|---|---|---|---|---|
| element 1 | 3.000E+03 | -1.851E-01 | -1.856E-01 | 100.3 |
| element 2 | 4.827E+02 | -2.345E-01 | -1.893E-01 | 80.7 |
| element 3 | 1.068E+03 | -7.332E-01 | -7.335E-01 | 100.0 |
| node 2 | -1.009E+00 | 1.567E-07 | 1.578E-07 | 100.7 |

Table 2 Sensitivity w.r.t. orient. angle of ③

| | Values | Sensitivity by FD | Sensitivity by DSA | Ratio(%) |
|---|---|---|---|---|
| element 1 | 3.000E+03 | 3.309E-01 | 3.300E-01 | 99.7 |
| element 2 | 4.827E+02 | 2.594E-01 | 2.600E-01 | 100.2 |
| element 3 | 1.068E+03 | 8.171E-01 | 8.176E-01 | 100.1 |
| node 2 | -1.009E+00 | -2.900E-09 | -1.229E-12 | 0.0 |

Even with employing a simple linear function for the design velocity, results are found excellent. In the formula by Twu and Choi [8], a special numerical discontinuity problem appeared due to the 2nd order derivative of design velocity, which was avoided by evaluating the so called corner terms in complicated way or employing cubic design velocity, both of which are not tractable in numerical view point. More elegant formulation is developed in this paper by employing cartesian description and introducing end term differences over each segment in general manner, which is easy to implement even with linear velocity functions. Furthermore consideration for an orientation sensitivity made in this paper gives more freedom in the optimization.

## 4. Examples of Shape Optimization

Problem 1: Tilted arch bridge

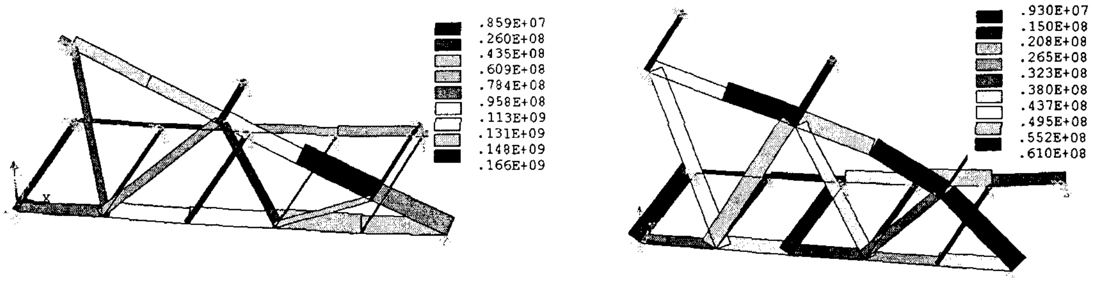
Optimization is carried out for the tilted arch bridge as shown in Fig 4. The shape of the arch is represented in this problem by the rational quadratic curve which is given by

$$H(\eta)=\frac{(1-\eta)^2 h_1 w_1 + 2\eta(1-\eta)h_2 w_2 + \eta^2 h_3 w_3}{(1-\eta)^2 w_1 + 2\eta(1-\eta) w_2 + \eta^2 w_3} \tag{17}$$

where $\eta$ is normalized parameter which runs from 0 to 1 along the bridge arch. Let $w_1, w_3 = 1$ and $h_3 = 0$, and choose $h_1, h_2$ and $w_2$ as the design parameters. Tilt angle $\alpha$ is added to the design parameters. The orientation angle of the cross section of the arch and its base beam is made aligned with $\alpha$. Self weight is applied in this problem. The objective is to minimize maximum stress of the structure by adjusting the tilt angle and shape of the arch while maintaining the total length of the beam at a fixed value.

Table 3. Initial and optimum solution of tilted arch bridge

| | | Design parameter (normalized) | | | | Max. stress (Mpa) | Number of iteration |
|---|---|---|---|---|---|---|---|
| | | Tilt angle α | h1 | h2 | w2 | | |
| Case1 | Initial | 0.2000 | 2.0000 | 1.0000 | 1.0000 | 165.7 | 17 |
| | Optimum | 0.7010 | 1.8050 | 1.4270 | 1.7340 | 61.1 | |
| Case2 | Initial | 0.9000 | 2.0000 | 1.0000 | 1.0000 | 76.4 | 18 |
| | Optimum | 0.7103 | 1.7530 | 1.4200 | 1.8220 | 61.1 | |




(a) (b)

Figure 4. Stress plot at initial and optimum design of tilted arch bridge (unit in Pa)
(a) Initial design with $\alpha$=20° (b) optimum design where $\alpha = 70.1°$

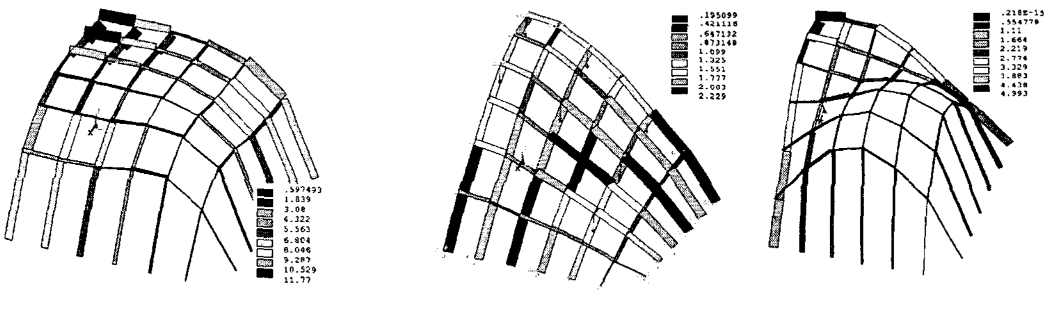
Stresses are calculated at each beam segment of the structure. Two initial designs are considered, of which the tilt angle is given by 20° and 90° respectively. Initial shape of the arch is given by straight line. IDESIGN [11] is used as optimization routine. Convergence parameter and maximum constraint violence are 1.e−2. The result is given in the Table 3, which shows the similar solution is obtained from different initial designs. In Figure 4, stress plots at the initial with tilt angle 20° and optimum design are shown, in which the color and thickness denote the magnitude of the stress.

Problem 2: Archgrid structure

The problem considered by Rozvany and Prager[10] is considered for the optimization study. In this problem, First case is to minimize maximum stress under the given total length of the structure. Second case is to maximize the volume inside the archgrid structure under the allowable stress with 5 Mpa. The initial and optimum solutions are given in Figure 5. Optimization results are given in Table 4, in which value of the volume is normalized with respect to the initial one, which means volume is increased by 77% at the optimum.

Table 4. Initial and optimum solution of archgrid problem

| | | Volume inside archgrid | Max. stress (Mpa) | Number of iteration |
|---|---|---|---|---|
| Case1 | Initial | - | 11.77 | 16 |
| | Optimum | - | 2.23 | |
| Case2 | Initial | 1.000 | 11.77 | 13 |
| | Optimum | 1.770 | 5.00 | |



(a) (b) (c)

Figure 5. Stress plot at initial and optimum shape of archgrid problem
(a) Initial design (b) optimum design for case 1 (c) optimum design for case 2

References
1. Adelman HM, Haftka RT. Sensitivity analysis for discrete structural systems. AIAA Journal 1986; 24(5):831-900.
2. Haug EJ, Choi KK, Komkov V. *Design Sensitivity Analysis of Structural Systems.* Academic Press : New York, 1986.
3. Topping BHV. Shape optimization of skeletal structures: A review. ASCE Journal of Structural Engineering 1983; 109(8):1933-1952.
4. Saka MP, Attili B. Shape optimization of space trusses. in Topping BHV. (ed.), Proceedings of International Conference on the Design and Construction of Non-conventional Structures. Civil-Comp Press : London, 1987; 115-121.
5. Hansen SR, Vanderplaats G.N. An approximation method for configuration optimization of trusses. AIAA/ASME/ASCE/AHS 29th Structures, Structural Dynamics and Materials Conference, Part 3. 1988; No. 88-2432.
6. Dems K, Mroz Z. A variational approach to sensitivity analysis and structural optimization of plane arches, Mechanics of Structures and Machines, 1987; 15;297-321.
7. Twu S, Choi KK. Configuration design sensitivity analysis of built-up structures part I: theory. International Journal for Numerical Methods in Engineering 1993; 35:1127-1150.
8. Twu S, Choi KK. Configuration design sensitivity analysis of built-up structures part II: numerical method. International Journal for Numerical Methods in Engineering 1993; 36:4201-4222.
9. Park YH, Choi KK. Configuration design sensitivity analysis of nonlinear structural systems with elastic material. Mechanics of Structures and Machines 1996; 24(2):217-255.
10. Rozvany G.IN, Prager W. A new class of structural optimization problems: optimal archgrid. Computer Methods in Applied Mechanics and Engineering 1979; 19:127-150.
11. Arora, JS, Tseng, CH. User's Manual for IDESIGN: version 3.5. Optimal Design Lab., Univ. of Iowa, 1986.