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Some properties of type-2 fuzzy set-valued mappings

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Abstract

In this paper we introduce the concepts of type-2 fuzzy set-valued mappings and quasi-convex fuzzy mappings on $L-L$ fuzzy numbers and discuss some properties of these mappings.

Key-words: fuzzy numbers; L-L fuzzy numbers; fuzzy mapping; quasi-convex

1. Preliminaries and definitions

Let X be a finite set. A fuzzy set A in X is defined by $A=\{(x, \mu_A(x)) | x \in X\}$, where $\mu_A : X \to [0,1]$ is the membership function of A. When $\mu_A(x)$ becomes a fuzzy set, A becomes a type-2 fuzzy set.

Now, since a type-2 fuzzy set is obtained by assigning fuzzy membership values to elements of X, we can extend the set-theoretic operations of ordinary fuzzy set theory to allow them to deal with fuzzy grades of membership.

We will use the following concept of fuzzy number. Let R be the real numbers and let $[0,1]$ be the unit interval in R. Let $F(R)$ denote the set of all fuzzy sets in R.

Definition 1.1 [1,2] A fuzzy set $A \in F(R)$ is called a fuzzy number if

(1) A is normal; i. e., there exists $x_0 \in R$ such that $\mu_A(x_0)=1$,

(2) A is convex; i. e., $\mu_A(\lambda x+(1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$ for all $x, y \in R$ and $\lambda \in [0,1]$,

(3) For any $\alpha \in [0,1]$, α is a closed interval and $cl(A_0) = cl$ $\mu_A(x) > 0)$ is compact. Here, $cl(A_0)$ is the closure of A_0.

Definition 1.2 [2] A fuzzy number M is said to be $L-R$ fuzzy number, $M=(m, \alpha, \beta)_{LR}$ if its membership function is defined by

(i) when $\alpha>0$ and $\beta>0$,

$$\mu_{M(x)} = \begin{cases} L(\frac{m-x}{\alpha}) & \text{for } m-\alpha \leq x \leq m \leq 1, x \geq 0 \\ R(\frac{x-m}{\beta}) & \text{for } m+\beta \geq x \geq m \geq 0, x \leq 1 \\ 0 & else \end{cases}$$

(ii) when $\alpha=0$ and $\beta>0$,

$$\mu_{M(x)} = \begin{cases} R(\frac{x-m}{\beta}) & \text{for } m+\beta \geq x \geq m \geq 0, x \leq 1 \\ 0 & else \end{cases}$$

(iii) when $\alpha>0$ and $\beta=0$,

$$\mu_{M(x)}=\begin{cases} L(\frac{m-x}{\alpha}) & \text{for } m-\alpha\le x\le m\le 1, x\ge 0 \\ 0 & else \end{cases}$$

(iv) when $\alpha=0$ and $\beta=0$,

$$\mu_{M(x)}=\begin{cases} 1 & \text{for } x=m \\ 0 & else \end{cases},$$

where α and β are called the left and right spreads of an $L-R$ fuzzy number M, respectively, and L and R are strictly decreasing continuous functions from $[0,1]$ to $[0,1]$ such that $L(0)=R(0)=1$ and $L(1)=R(1)=0$. In this case, L and R is called the left and the right shape function, respectively. A_{LR} will stand for the class of all $L-R$ fuzzy numbers of $[0,1]$.

Definition 1.3 [2] Let $M_x=(m_x, \alpha_x, \beta_x)_{LR}$ and $M_y=(m_y, \gamma_y, \delta_y)_{LR}$ be elements to A_{LR}.

Then $\max^{\sim}(M_x, M_y)$, $\min^{\sim}(M_x, M_y)$ are defined by

$$\max{}^{\sim}(M_x, M_y)=(m_x\vee m_y,\ \alpha_x\wedge\gamma_y,\ \beta_x\vee\delta_y)_{LR},$$

$$\min{}^{\sim}(M_x, M_y)=(m_x\wedge m_y,\ \alpha_x\vee\gamma_y,\ \beta_x\wedge\delta_y)_{LR},$$

where $x\vee y$ and $x\wedge y$ are max x, y and min x, y, respectively.

We note that max$^{\sim}$ and min$^{\sim}$ are commtative and associative operations; they are mutually distributive.

Theorem 1.4 [2] If $M=(m, \alpha, \beta)_{LR}$ and $N=(n, \gamma, \delta)_{RL}$ be elements to A_{LR} and A_{RL}, respectively, then

$$M\ominus N=(m-n,\ \alpha+\gamma,\ \beta+\alpha)_{LR}.$$

By using the theorem 1.4, we can obtain the following complement of $L-R$ type-2 fuzzy numbers.

Definition 1.5 [2] Let $M=(m, \alpha, \beta)_{LR}$ and $M^1=(1,0,0)_{RL}$ be elements to A_{LR} and A_{LR}, respectively. The complemented M^* of M is defined by $M^*\equiv M^1\ominus M=(1-m,\ \beta,\ \alpha)_{RL}$.

2. Main results

Let X and Y be finite sets. We consider that A_{LL} is the class of all $L-L$ fuzzy numbers of $[0,1]$.

The collection of type-2 fuzzy set-valued mappings of a set X is denoted by $F_2(X)$, i.e., $M\in F_2(X) \Leftrightarrow M: X\to A_{LL}$ by $M(x)=M_x$.

Definition 2.1 We say that ψ is a type-2 fuzzy set-valued mapping on $X\times Y$ if (1) $\psi: X\times Y\to A_{LL}$ by $\psi(x,y)=M_{xy}\in A_{LL}$, $\forall(x,y)\in X\times Y$ (2) $\forall x\in X$, there exists $y\in Y$ such that $\max^{\sim}_x(\psi(x,y))=(1,0,0)_{LL}=M^1$.

Definition 2.2 Let ψ be a type-2 fuzzy set-valued mapping on $X\times Y$. T_ψ is called *the inverse image operator associated with* ψ iff $\forall M\in F_2(Y)$, $\forall x\in X$,

$$(T_\psi M)(x)=\max{}^{\sim}_x(\min{}^{\sim}_y(\psi(x,y), M_y)).$$

From the definition 2.2, it is ease to show that $T_\psi: F_2(Y)\to F_2(X)$ is a mapping from type-2 fuzzy sets of Y to type-2 fuzzy sets of X.

Definition 2.3 [2] Let $M_x=(m_x, \alpha_x, \beta_x)_{LL}$ and $N_x=(n_x, \gamma_x, \delta_x)_{LL}$ be elements of A_{LL}.

Then, we define the order $\le$ of M_x and N_x; $M_x\le N_x$ if and only if $m_x\le n_x$, $\alpha_x\ge\gamma_x$, and $\beta_x\le\delta_x$.

Definition 2.4 [2] Let $M, N: X\to A_{LL}$ be type-2 fuzzy set-valued mappings of a set X. Then we define the order $\le$ of M and N; $M\le N$ iff $M_x\le N_x$, $\forall x\in X$.

Theorem 2.5 [2] Let $M=(m,\alpha,\beta)_{LL}$ and $N=(n,\gamma,\delta)_{LL}$ be elements of A_{LL}. Then, we have $\max^{\sim}(M,N)=N$, $\min^{\sim}(M,N)=M$ if and only if $m\leq n$, $\alpha\geq\gamma$, and $\beta\leq\delta$.

Proposition 2.6 Let T_ψ be the inverse image operator associated with ψ and $M^0=(0,0,0)_{LL}$. Then we have

(1) $(T_\psi M^0)(x)\leq M^0$,

(2) $(T_\psi M^1)(x)\leq M^1$,

(3) T_ψ is order preserving,

(4) $(T_\psi M^{0*})(x)\leq M^1$,

(5) $(T_\psi M^{1*})(x)\leq M^0$.

We note that if $M,N\in F_2(Y)$, then $\max^{\sim}(M,N)$ means $\max^{\sim}(M,N)(y)=\max^{\sim}(M_y,N_y)$ for all $y\in Y$ and $\max^{\sim}(T_\psi M, T_\psi N)$ means $\max^{\sim}(T_\psi M, T_\psi N)(x)=\max^{\sim}(T_\psi M(x), T_\psi N(x))$ for all $x\in X$.

Proposition 2.7 Let M and N be elements of A_{LL} and let T_ψ be the inverse image operator associated with ψ. Then we have

$$T_\psi(\max^{\sim}(M,N))=\max^{\sim}(T_\psi M, T_\psi N).$$

Proposition 2.8 Let M and N be elements of A_{LL} and let T_ψ be the inverse image operator associated with ψ. Then we have $T_\psi(\min^{\sim}(M,N))\leq\min^{\sim}(T_\psi M, T_\psi N)$ with equality hold if $M\leq N$ or $M\geq N$.

Now, we introduce the concept of quasi-convex and quasi-concave fuzzy mappings and present some properties of theses fuzzy mappings.

Definition 2.9 Let $L,M,$ and N be elements of A_{LL}. A fuzzy mapping $S:F_2(Y)\to F_2(X)$ is said to be quasi-convex, if $S(M)\leq\max^{\sim}S(L),S(N)$ whenever $L\leq M\leq N$.

Definition 2.10 Let $L,M,$ and N be elements of A_{LL}. A fuzzy mapping $S:F_2(Y)\to F_2(X)$ is said to be quasi-concave, if

$S(M)\geq\min^{\sim}S(L),S(N)$ whenever $L\leq M\leq N$.

Proposition 2.11 For any type-2 fuzzy set-valued mapping ψ, T_ψ is quasi-convex fuzzy mapping.

Proof. Let L_y,M_y,N_y be elements of A_{LL} and let $L_y\leq M_y\leq N_y$. By proposition 2.6 (3) and theorem 2.5, we have $\max^{\sim}(T_\psi L, T_\psi N)\leq\max^{\sim}(T_\psi L, T_\psi M)=T_\psi M$. □

Proposition 2.12 For any type-2 fuzzy set-valued mapping ψ, T_ψ is quasi-concave fuzzy mapping.

Proof. The proof of this proposition is quite similar to that of proposition 2.11.

Proposition 2.13 Let ψ be a type-2 fuzzy set-valued mapping on $X\times Y$. If S_ψ is any inverse image operator associated with ψ and T_ψ is quasi-convex fuzzy mapping with respect to, then $S_\psi\circ T_\psi$, the composition of S_ψ and T_ψ, is quasi-convex fuzzy mapping.

Proof. Let $L_y\leq M_y\leq N_y$ in $F_2(Y)$. Since T_ψ is quasi-convex fuzzy mapping, by using the proposition 2.7, we have

$$(S_\psi\circ T_\psi)(M_y)=S_\psi(T_\psi(M_y))$$
$$\leq S_\psi(\max^{\sim}(T_\psi(L_y), T_\psi(N_y)))$$
$$=\max^{\sim}(S_\psi(T_\psi(L_y)), S_\psi(T_\psi(N_y))).$$

Thus $S_\psi\circ T_\psi$ is quasi-convex fuzzy mapping. □

Proposition 2.14 Let ψ be a type-2 fuzzy set-valued mapping on $X\times Y$. If S_ψ and T_ψ are quasi-convex fuzzy mappings, then

$S_\psi \circ T_\psi$ is quasi-convex fuzzy mapping.

Definition 2.15 Let M and N be elements of A_{LL}. A fuzzy mapping $S: F_2(Y) \rightarrow F_2(X)$ is said to be convex, if $S(\lambda M + (1-\lambda)N) \leq \lambda S(M) + (1-\lambda)S(N)$ for all $\lambda \in [0,1]$.

Proposition 2.16 Let ψ be a type-2 fuzzy set-valued mapping on $X \times Y$. If S_ψ and T_ψ are convex fuzzy mappings, then $S_\psi \circ T_\psi$ is convex fuzzy mapping.

Proposition 2.17 For any type-2 fuzzy set-valued mapping ψ and ϕ, if $\max^{\sim}(\psi(x,y)) = \max^{\sim}(\phi(x,y))$, then $T_\psi = T_\phi$.

Proof. Suppose that $\psi = \phi$. Then, this property is trivial. Let $\psi(x,y) \neq \phi(x,y)$ be type-2 fuzzy set-valued mappings on $X \times Y$.

Let $\psi(x,y) = M_{xy} = (m_{xy}, \alpha_{xy}, \beta_{xy})_{LL}$ and $\phi(x,y) = N_{xy} = (n_{xy}, \gamma_{xy}, \delta_{xy})_{LL}$. By hypothesis, $\max^{\sim}(\psi(x,y)) = (\bigvee_x m_{xy}, \bigwedge_x \alpha_{xy}, \bigvee_x \beta_{xy})_{LL} = \max^{\sim}(\phi(x,y)) = (\bigvee_x n_{xy}, \bigwedge_x \gamma_{xy}, \bigvee_x \delta_{xy})_{LL}$, we have

$$(T_\psi M)(x) = \max{}_x^{\sim}(\min{}_y^{\sim}(\psi(x,y), M_y))$$
$$= \max{}_x^{\sim}(\min{}_y^{\sim}(M_{xy}, M_y))$$
$$= \max{}_x^{\sim}(\min{}_y^{\sim}((m_{xy}, \alpha_{xy}, \beta_{xy})_{LL}, (m_y, \alpha_y, \beta_y)_{LL}))$$
$$= \max{}_x^{\sim}((m_{xy} \wedge m_y), (\alpha_{xy} \vee \alpha_y), (\beta_{xy} \wedge \beta_y))_{LL}$$
$$= (\bigvee_x (m_{xy} \wedge m_y), \bigwedge_x (\alpha_{xy} \vee \alpha_y), \bigvee_x (\beta_{xy} \wedge \beta_y))_{LL}$$
$$= (\bigvee_x m_{xy} \wedge \bigvee_x m_y), (\bigwedge_x \alpha_{xy} \vee \bigwedge_x \alpha_y), (\bigvee_x \beta_{xy} \wedge \bigvee_x \beta_y))_{LL}$$
$$= (\bigvee_x n_{xy} \wedge \bigvee_x m_y), (\bigwedge_x \gamma_{xy} \vee \bigwedge_x \alpha_y), (\bigvee_x \delta_{xy} \wedge \bigvee_x \beta_y))_{LL}$$
$$= \max{}_x^{\sim}(\min{}_y^{\sim}((n_{xy}, \gamma_{xy}, \delta_{xy})_{LL}, (m_y, \alpha_y, \beta_y)_{LL}))$$
$$= \max{}_x^{\sim}(\min{}_y^{\sim}(N_{xy}, M_y))$$
$$= \max{}_x^{\sim}(\min{}_y^{\sim}(\phi(x,y), M_y)) = (T_\phi M)(x)$$
$$\forall M_y = (m_y, \alpha_y, \beta_y) \in A_{LL}.$$

Thus $T_\psi = T_\phi$. □

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