Proceedings of KFIS 2002 Spring Conference, 2002. 5. 25

집합치 쇼케이적분과 수렴성 정리에 관한 연구(II)

On set-valued Choquet integrals and convergence theorems(II)

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<개요>

이 논문에서는 구간 수의 값을 갖는 함수들의 쇼케이적분을 생각하고자 한다. 이러한 구간 수의 값을 갖는 함수들의 성질들을 조사하여 오토연속인 퍼지측도에 관련된 쇼케이적분에 대한 수렴성 정리를 증명한다.

1. Introduction

It is well-known that closed set-valued functions had been used repeatedly in many papers [1,2,4,5,6,7,8]. Jang et al. [6,8] studied closed set-valued Choquet integrals and convergence theorems under some sufficient conditions, for examples; (i) convergence theorems for monotone convergent sequences of Choquet integrably bounded closed set-valued functions(see [6]), (ii) covergence theorems for the upper limit and the lower limit of a sequence of Choquet integrably bounded closed set-valued functions (see[8]).

The aim of this paper is to prove convergence theorem for convergent sequences of Choquet integrably bounded interval number-valued functions in the metric Δ_S (see Definition 3.4).

2. Definitions and preliminaries

Definition 2.1 [7,9] (1) A *fuzzy measure* on a measurable space (X, $\mathscr{B}$) is an extended real-valued function

$\mu : \mathscr{B} \rightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset)=0$

(ii) $\mu(A) \leq \mu(B)$,

whenever $A, B \in \mathscr{B}$, $A \subset B$.

(2) A fuzzy measure μ is said to be autocontinuous from above[resp., below] if $\mu(A \cup B_n) \rightarrow \mu(A)$ [resp., $\mu(A \sim B_n) \rightarrow \mu(A)$] whenever $A \in \mathscr{B}$, $\{B_n\} \subset \mathscr{B}$ and $\mu(B_n) \rightarrow 0$.

(3) If μ is autocontinouos both from above and from below, it is said to be autocontinouos.

Recall that a function $f: X \rightarrow [0, \infty)$ is said to be measurable if $\{x | f(x) > \alpha\} \in \mathscr{B}$ for all $\alpha \in (-\infty, \infty)$.

Definition 2.2 [9] (1) A sequence $\{f_n\}$ of measurable functions is said to converge to f in measure, in symbols $f_n \rightarrow_M f$ if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mu(\{x | |f_{n(x)} - f(x)| > \varepsilon\}) = 0.$$

(2) A sequence $\{f_n\}$of measurable functions is said to converge to f in distribution, in symbols $f_n \rightarrow_D f$ if for every $\varepsilon > 0$,

$\lim_{n \rightarrow \infty} \mu_{f_n}(r) = \mu_f(r)$ $e.c.$, where $\mu_f(r) = \mu(\{x | f(x) > r\})$ and "e.c." stands for "except at most countably many values of r".

Definition 2.3 [9] (1) The Choquet integral of a measurable function f with respect to a fuzzy measure μ is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r)\, dr$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet integral of f can be defined and its value is finite.

Throughout the paper, R^+will denote the interval $[0, \infty)$,

$I(R^+) = \{[a, b] | a, b \in R^+ \text{and } a \leq b\}$. Then a element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a, b], [c, d] \in I(R^+)$ and $k \in R^+$,

$[a, b] + [c, d] = [a + c, b + d]$,

$[a, b] \cdot [c, d] = [a \cdot c,\ b \cdot d]$,

$k[a, b] = [ka, kb]$,

$[a, b] \leq [c, d] \Leftrightarrow a \leq c$ and $b \leq d$,

Then $(I(R^+), d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_{H(A, B)} = \max\{\sup_{x \in A} \inf_{y \in B} |x - y|,\ \sup_{y \in B} \inf_{x \in A} |x - y|\}$$

for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

Proposition 2.4 For each pair $[a, b], [c, d] \in I(R^+)$,

$d_H([a, b], [c, d]) = \max\{|a - c|, |b - d|\}$.

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a closed set-valued function $F : X \rightarrow C(R^+) \setminus \{\emptyset\}$ and an interval number-valued function

$F: X \to I(R^{+}) \setminus \{\emptyset\}$. We denote that $d_H - \lim_{n\to\infty} A_n = A$ if and only if

$\lim_{n\to\infty} d_H(A_n, A) = 0$, where $A \in I(R^{+})$

and $\{A_n\} \subset I(R^{+})$.

We say $f: X \to R^{+}$ is in $L_c^1(\mu)$ if and only if f is measurable and $(C)\int f d\mu < \infty$. We note that "$x \in X$ $\mu - a.e.$" stands for "$x \in X$ μ-almost everywhere". The property $P(x)$ holds for $x \in X$ $\mu - a.e.$ means that there is a measurable set A such that $\mu(A) = 0$ and the property $P(x)$ holds for all $x \in A^c$, where A^c is the complement of A.

Definition 2.5 [5,6] (1) Let F be a closed set-valued function and $A \in \mathscr{B}$. The Choquet integral of F on A is defined by

$$(C)\int_A F d\mu = \{(C)\int_A f d\mu \mid f \in S_c(F)\}.$$

where $S_c(F)$ is the family of $\mu - a.e.$ Choquet integrable selections of F, that is,

$$S_c(F) = \{f \in L_c^1(\mu) \mid f(x) \in F(x) \quad x \in X \ \mu - a.e.\}$$

(2) A closed set-valued function F is said to be Choquet integrable if $(C)\int F d\mu \neq \emptyset$.

(3) A closed set-valued function F is said to be Choquet integrably bounded if there is a function $g \in L_c^1(\mu)$ such that

$$|F(x)| = \sup_{r \in F(x)} |r| \leq g(x) \quad \text{for all } x \in X.$$

Instead of $(C)\int_X F d\mu$, we will write $(C)\int F d\mu$. Let us discuss some basic properties of measurable closed set-valued functions. Since $R^{+} = [0, \infty)$ is a complete separable metric space in the usual topology, using Theorem 8.1.3([1])and Theorem 1.0(2^0)([5]), we have the following theorem.

Theorem 2.6 [1,4] A closed set-valued function F is measurable if and only if there exists a sequence of measurable selections $\{f_n\}$ of F such that

$$F(x) = \text{cl}\{f_n(x)\} \quad \text{for all } x \in X.$$

3. Main results

Since $(X, \mathscr{B})$ is a measurable space and R^{+} is a separable metric space, Theorem 1.0(2^0)([4]) implies the following theorem. Recall that a measurable closed set-valued function is said to be convex-valued if $F(x)$ is convex for all $x \in X$ and that a set A is an interval number if and only if it is closed and convex.

Theorem 3.1 If F is a measurable closed set-valued function and Choquet integrably bounded, then there exists a sequence $\{f_n\}$ of Choquet integrable functions $f_n: X \to R^{+}$ such that $F(x) = \text{cl}\{f_n(x)\}$ for all $x \in X$.

Theorem 3.2 If F is a measurable closed set-valued function and Choquet integrably bounded and if we define $f^*(x) = \sup\{r \mid r \in F(x)\}$ and $f_*(x) = \inf\{r \mid r \in F(x)\}$ for all $x \in X$, then f^* and f_* are Choquet integrable selections of F.

Assumption (A) For each pair $f, g \in S_c(F)$, there exists $h \in S_c(F)$ such that $f \sim h$ and $(C)\int g d\mu = (C)\int h d\mu$.

We consider the following classes of interval number-valued functions;

$$\mathfrak{I} = \{F | F : X \to I(R^+) \text{ is measurable and Choquet integrably bounded}\}$$

and

$$\mathfrak{I}_1 = \{F \in \mathfrak{I} | F \text{ is convex-valued and satisfies the assumption(A)}\}.$$

Theorem 3.3 If $F \in \mathfrak{I}_1$, then we have

(1) $cF \in \mathfrak{I}_1$ for all $c \in R^+$,

(2) $(C)\int F d\mu$ is convex,

(3) $(C)\int F d\mu = [(C)\int f_* d\mu, (C)\int f^* d\mu]$.

We consider a function $\triangle_S$ on $\mathfrak{I}_1$ defined by

$$\triangle_S(F, G) = \sup_{x \in X} d_H(F(x), G(x))$$

for all $F, G \in \mathfrak{I}_1$. Then, it is easily to show that $\triangle_S$ is a metric on $\mathfrak{I}_1$.

Definition 3.4 Let $F \in \mathfrak{I}_1$. A sequence $\{F_n\} \subset \mathfrak{I}_1$ converges to F in the metric $\triangle_S$, in symbols, $F_n \to_{\triangle_S} F$ if $\lim_{n\to\infty} \triangle_S(F_n, F) = 0$.

Theorem 3.5(Convergence Theorem)} Let $F, G, H \in \mathfrak{I}_1$ and $\{F_n\}$ be a sequence in $\mathfrak{I}_1$. If a fuzzy measure μ is autocontinuous and if $F_n \to_{\triangle_S} F$ and $G \le F_n \le H$, then we have

$$d_H\text{-}\lim_{n\to\infty}(C)\int F_n d\mu = (C)\int F d\mu.$$

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