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구체무단변속기의 비선형 피드백제어기 설계

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Shifting Controller Design via Exact Feedback Linearization of a Spherical Continuously Variable Transmission

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Key Words : Spherical continuously variable transmission; shifting controller; input-state feedback linearization.

Abstract

The spherical CVT, intended to overcome some of the limitations of existing CVT designs, is marked by its simple kinematic design, improved efficiency of the shift actuator, and IVT characteristics, *i.e.,* the ability of smooth transition between the forward, neutral, and reverse states without the need for any brakes or clutches. And it has been promised much possibility of energy savings and various applications for small power capacity machinery. Due to the nonlinearity of the spherical CVT shifting dynamics, however the original open-loop system is inherently unstable. Hence a feedback controller is necessary to make the system stable and to achieve effective tracking performance. To do this, we designed a feedback controller that cancels nonlinearities and transforms the original nonlinear system dynamics into a stable and controllable linear one, based on the input-state linearization method.

1. Introduction

The most important role of the shifting controller for continuously variable transmissions (CVTs) is the realization of the target gear ratio, which is directly related to the input/output ratio of power. When the shifting command for a certain gear ratio is given, the shifting system must be stabilized so as to realize the demanded gear ratio with the desired performance (*e.g.,* little shifting effort, short settling time, *etc.*). The shifting command of a CVT can be either a final value or a trajectory of the target gear ratio. According to the shifting command, the shifting controller design task is denoted as *stabilizer* (or *regulator*) design for the former and *tracker* (or *servo*) design for the latter.

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In control theory, a basic problem is how to use feedback in order to modify the original internal dynamics of a controlled plant so as to achieve some prescribed behavior. In particular, feedback may be used for the purpose of imposing, on the associated closed-loop system, the (unforced) behavior of some prescribed autonomous linear system. When the plant is modeled as a linear time-invariant system, this is known as the problem of pole placement, while in the more general case of a nonlinear model, this is known as the problem of *feedback linearization* [1]-[4].

Feedback linearization is an approach to nonlinear control design which has attracted a great deal of research interest in recent years. The central idea is to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied. This differs entirely from conventional linearization in that feedback linearization is achieved by exact state transformations and feedback, rather than by linear approximations of the dynamics (*i.e.*, Jacobian linearization).

The shifting system of the spherical CVT (S-CVT) has second-order nonlinear dynamics, and the original open-loop system reveals unstable characteristics. In order to cancel nonlinearities of the S-CVT shifting system, and to make it stable and have good tracking characteristics, we develop a feedback controller based on the exact feedback linearization method in this paper. We first review the structure and operating principles of the S-CVT briefly in Section 2. In Section 3 we investigate the instability of the original shifting system using Lyapunov's indirect method. Section 4 address the input-state feedback controller design of the S-CVT shifting system. Finally, we investigate the stabilizing and tracking performance of the dedicated shifting controller by numerical simulation.

2. Spherical CVT

In our previous work [5], we have presented the design and analysis of the S-CVT. The S-CVT, intended to overcome some of the limitations of existing CVT designs, is marked by its simple kinematic design, improved efficiency of the shift actuator, and IVT characteristics, *i.e.*, the ability of smooth transition between the forward, neutral, and reverse states without the need for any brakes or clutches.

The S-CVT is composed of three pairs of input and output discs, variators, and a sphere (see Figure 1). The input discs are connected to the power source, while the output discs are connected to the output shafts. The sphere, which is the main component of the S-CVT, transmits power from the input discs to the output discs via rolling resistance between the discs and the sphere. The variators, which are connected to the shifting controller, are in contact with the sphere like the discs, and constrain the direction of rotation of the sphere to be tangent to the rotational axis of the variator.

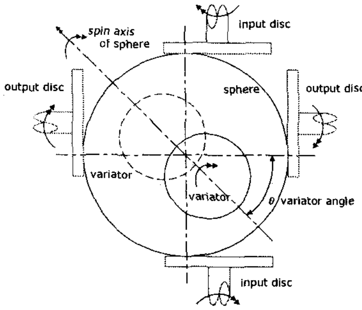


Fig. 1 Structure of the S-CVT.

By varying the axis of rotation of the sphere, it is in turn possible to vary the radius of rotation of the contact point between the input disc and the sphere, R_i, as well as the radius of rotation of the contact point between the output disc and the sphere, R_o (see Figure 2). In this way the speed-torque ratio of the S-CVT can be adjusted.

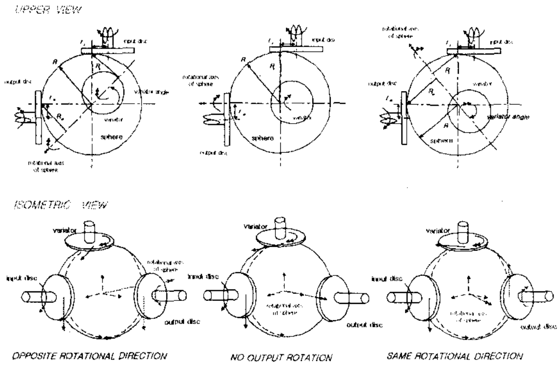


Fig. 2 Operating principles of the S-CVT.

Figure 2 shows the various alignments of the variator for the forward, neutral, and reverse states of the output shaft of the S-CVT. The neutral state, which corresponds to zero rotation of the output disc, is achieved when R_o becomes zero. As apparent from the figure, the forward, neutral, and reverse states can all be achieved by smoothly manipulating the variator alignment, without the need for any additional clutches or brakes.

Assuming roll contact without slip, the speed and torque ratio between the input and output discs is related to the variator angle by the following relations:

$$\frac{\omega_{out}}{\omega_{in}} = \frac{r_i}{r_o}\tan\theta, \quad \frac{T_{out}}{T_{in}} = \frac{r_o}{r_i}\cot\theta \qquad (1)$$

where θ is the angular displacement of the variator, ω_{in} and ω_{out} are the respective angular velocities of the input and output shafts, T_{in} and T_{out} are the respective input and output torques, and r_i and r_o are the respective radii of the contact points of the input and output discs (see Figure 2). Although ideally an infinite torque ratio is possible with the S-CVT as seen in Equation (1), in practice there is a limit to the torque that can be transmitted because power transmission occurs from rolling resistance of metal on metal.

Using the force analysis of the sphere in Figure 3, the dynamic equations of the S-CVT can be stated as follows (see more details in our previous work [5]);

$$\begin{bmatrix} 2\dfrac{(I_a + I_v + m\varepsilon^2)}{\varepsilon} & 2\dfrac{RI_v}{\varepsilon^2} \\ 2\dfrac{I_v}{\varepsilon} & \dfrac{I_s}{R} + 2\dfrac{RI_v}{\varepsilon^2} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} F_s \\ F_i\cos\theta - F_o\sin\theta \end{bmatrix} \qquad (2)$$

The nomenclature adopted in the dynamics of the S-CVT represents

I_a = moment of inertia of the shifting actuator shaft and connecting rod [$kg{\cdot}m^2$],
I_v = moment of inertia of the variator [$kg{\cdot}m^2$],
I_s = moment of inertia of the sphere [$kg{\cdot}m^2$],
m = mass of the variator [kg],
ε = eccentric distance between the centers of the shifting actuator shaft and variator [m],
R = sphere radius [m],
F_s = shifting force [N],
F_i = driving force delivered from the input discs [N],
F_o = Reaction force caused by the output discs connected to the load torque [N],
ω = spinning rate of the sphere [rad/sec].

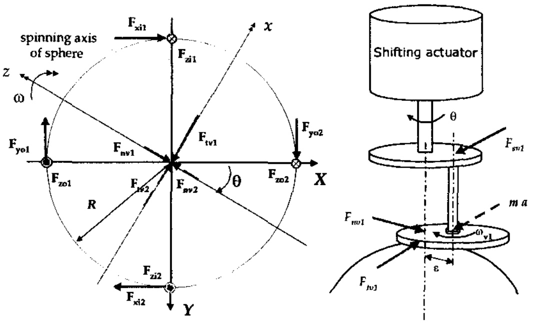


Fig. 3 Free body diagrams of the S-CVT.

3. Stability of S-CVT Shifting System

Lyapunov's (indirect) linearization method is involved with the local stability of a nonlinear system. It is a formalization of the intuition that a nonlinear system should behave similarly to its linearized approximation for small range motions. Because all physical systems are inherently nonlinear, Lyapunov's linearization method serves as the fundamental justification of using linear control techniques in practice, *i.e.*, that stable design by linear control guarantees the stability of the original physical system *locally*.

To determine the stability of the S-CVT shifting system, we first restate the shifting dynamics in Equation (2) into state-space form. Here we replace the state x_3 (the rotational speed of sphere) by a matrix transformation, because it does not affect the shifting dynamics. Letting $x_1 = \theta$, $x_2 = \dot{\theta}$ be the states, the corresponding state-space equation is the following second-order state equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{D}\{a_{12}\sqrt{F_i^2+F_o^2}\sin(x_1-\varphi)+a_{22}F_s\} \end{aligned} \qquad (3)$$

where

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2\frac{(I_a+I_v+m\varepsilon^2)}{\varepsilon} & 2\frac{RI_v}{\varepsilon^2} \\ 2\frac{I_v}{\varepsilon} & \frac{I_s}{R}+2\frac{RI_v}{\varepsilon^2} \end{bmatrix}, \; D = a_{11}a_{22} - a_{12}a_{21}$$

and $\varphi = \tan^{-1}(\frac{F_i}{F_o})$.

Considering D is always larger than zero, the equilibrium point is given by

$$x_1^* = \phi = \tan^{-1}\frac{F_i}{F_o},\; x_2^* = 0,\; F_s^* = 0.$$

We can say that the equilibrium point of interest is $\mathbf{x}^* = (\phi, 0)$. Physically, this point corresponds to the steady state of the shifting system in which shifting does not occur.

The Jacobian matrix J of the shifting system (3) linearized about the equilibrium point becomes

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ \frac{a_{12}}{D}\sqrt{F_i^2+F_o^2} & 0 \end{bmatrix},$$

and the eigenvalues of J are

$$\lambda_i = \pm\sqrt{\frac{a_{12}}{D}\sqrt{F_i^2+F_o^2}}.$$

Hence the linearized system is unstable, and therefore so is the shifting system of the S-CVT at this equilibrium point.

Physically this means that when the input and output force relation ($F_i\cos\theta = F_o\sin\theta$) is broken (*i.e.*, steady state is destroyed) by some disturbances from the input or output force, it must be followed by a change in variator angle (gear ratio) from the shifting actuator, or by a change in input force from the power source controller. In order to make the shifting system stable, one can conclude that an appropriate feedback controller is necessary. In the following sections, we will discuss the design of a feedback controller based on the exact feedback linearization method.

4. Shifting Controller Design via the Input-State Linearization

Based on the differential geometric definitions and theorems (see more details in references [1]-[4]), input-state linearization of the shifting system of the S-CVT has been performed via the following steps:

1. Construct the vector fields $g, ad_f g, \ldots, ad_f^{n-1} g$ for our system.

2. Check the controllable and involutive conditions.

3. Find the first new state z_1 from the relation between the vector fields;

$$\begin{aligned} \nabla z_1 \cdot ad_{\mathbf{f}}^k \mathbf{g} = 0 \quad k = 0,1,\ldots,n-2 \\ \nabla z_1 \cdot ad_{\mathbf{f}}^{n-1} \mathbf{g} \neq 0. \end{aligned} \tag{4}$$

4. Compute the diffeomorphism that transforms the state **x** into the new state **z**, $T(\mathbf{x}) = [z_1\ L_f z_1 \ldots L_f^{n-1} z_1]^T$, and the input transformation using

$$\alpha(x) = -\frac{L_f^n z_1}{L_g L_f^{n-1} z_1}, \quad \beta(x) = \frac{1}{L_g L_f^{n-1} z_1}. \tag{5}$$

where $L_f z_1$ is the Lie derivative of z_1 with respect to f, *i.e.*,

$$L_f z_1 = \nabla z_1 f .$$

First, we put the shifting system dynamics into the affine nonlinear control system form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot u$$

in order to obtain the corresponding vector fields **f** and **g**. Here we consider the shifting force F_s to be the control input u. Then **f** and **g** of the shifting dynamics can be written

$$\mathbf{f} = [x_2 \quad \frac{a_{12}}{D}\sqrt{F_i^2 + F_o^2}\sin(x_1 - \phi)]^T, \ \mathbf{g} = [0 \quad \frac{a_{22}}{D}]^T. \tag{6}$$

Knowing that the system order n = 2 and $\nabla\mathbf{g}$ = 0, the corresponding Lie bracket then becomes

$$ad_{\mathbf{f}}\mathbf{g} = \nabla\mathbf{g}\cdot\mathbf{f} - \nabla\mathbf{f}\cdot\mathbf{g}$$

$$= 0 - \begin{bmatrix} 0 & 1 \\ \frac{a_{12}}{D}\sqrt{F_i^2 + F_o^2}\cos(x_1 - \phi) & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{a_{22}}{D} \end{bmatrix} = \begin{bmatrix} -\frac{a_{22}}{D} & 0 \end{bmatrix}^T \tag{7}$$

4.1 Controllability and Linearizability

To determine the controllability of nonlinear systems, the controllability matrix C in the linear system is replaced by

$$[g \ \ ad_f g \ \ldots \ ad_f^{n-1} g].$$

In order to determine the controllability of the shifting system of the S-CVT, we investigate the rank of the controllability matrix using the results of Equations (6) and (7):

$$rank[\mathbf{g} \ \ ad_{\mathbf{f}}\mathbf{g}] = rank\begin{bmatrix} 0 & -\frac{a_{22}}{D} \\ \frac{a_{22}}{D} & 0 \end{bmatrix} = 2$$

Hence, we can say that the shifting system of the S-CVT is controllable. Furthermore, since the vector fields $\{\mathbf{g} \ \ ad_{\mathbf{f}}\mathbf{g}\}$ are constant (*i.e.*, its Lie derivatives are zero), they form an involutive set. Therefore the shifting system is *input-state linearizable*.

4.2 Input-State Linearization

Now we are ready to perform input-state linearization with the new states. First we find a diffeomorphism T(**x**) that can transform the original shifting dynamics into the linearized system. Using the results of Equation (4), the necessary conditions for the first state z_1 are

$$\frac{\partial z_1}{\partial x_1} \neq 0, \ \ \frac{\partial z_1}{\partial x_2} = 0.$$

Thus z_1 must be a function of x_1 only. Among the various candidates for z_1, the simplest solution is $z_1 = x_1 - \phi$. The other state can be obtained from z_1

$$z_2 = \nabla z_1 \mathbf{f} = x_2.$$

The corresponding diffeomorphism T(**x**) can be obtained as

$$\mathbf{z} = T(\mathbf{x}) = \begin{bmatrix} x_1 - \varphi \\ x_2 \end{bmatrix}.$$

Accordingly the input transformation in Equation (5) is

$$u = \frac{v - \nabla z_2 \mathbf{f}}{\nabla z_2 \mathbf{g}}$$

which can be written explicitly as

$$u = \frac{D}{a_{22}}\{v - \frac{a_{12}}{D}\sqrt{F_i^2 + F_o^2}\sin(x_1 - \phi)\}.$$

As a result of the above state and input transformations, we end up with the following set of linear equations

$$\begin{aligned} \dot{z}_1 = z_2, \ \ \dot{z}_2 = v \\ v = \frac{a_{12}}{D}\sqrt{F_i^2 + F_o^2}\sin(x_1 - \varphi) + \frac{a_{22}}{D}u \end{aligned} \tag{8}$$

5. Shifting Controller Design

By the above input-state linearization results, we now perform the shifting controller design which can stabilize the shifting system according to the shift command and track the demanded variator angle trajectory.

5.1 Stabilizing Controller Design

Since the new dynamics (8) is linear and controllable, it is well known that the linear state feedback control law

$$v = -k_1 z_1 - k_2 z_2$$

can guarantee asymptotic stability by selecting feedback gains k_1 and k_2 so as to satisfy the Hurwitz condition. The linearized system can be written

$$\ddot{z}_1 + k_2\dot{z}_1 + k_1 z_1 = 0 \tag{9}$$

5.2 Tracking Controller Design

For the case of the tracking problem, it is desired to have the variator angle θ track a prescribed trajectory θ_d. Then the input v is designed as

$$v = \ddot{z}_{1d} - k_1 e - k_2 \dot{e}$$

where $e = z_1 - z_{1d}$ and $z_{1d} = \theta_d - \phi$. Therefore, the tracking problem of linearized shifting dynamics transforms into the following error dynamics:

$$\ddot{e}+k_2\dot{e}+k_1 e=0 \qquad (10)$$

In order to guarantee asymptotic tracking performance of the shifting system, one may check whether the gain selection k_1, k_2 can satisfy the Hurwitz condition, similarly to the case of stabilizer design. The Hurwitz condition, however, offers only a set of inequalities for the feedback gains, and the gain selection within these boundaries must be achieved using other criteria.

5.3 Gain Selection

The resulting closed-loop dynamics of the shifting system (9), (10) can be viewed as the canonical form of a general second-order oscillation problem:

$$s^2+2\zeta\omega_n s+\omega_n^2=0\,.$$

Hence one can give physical meaning to the feedback gains as the respective damping ratio ζ and the natural frequency ω_n. The relation between the feedback gains and ζ, ω_n are simply

$$k_1=\omega_n^2,\quad k_2=2\zeta\omega_n\,.$$

Therefore, we can deduce the relation of the gains k_1, k_2 as follows:

$$k_2=2\zeta\sqrt{k_1}$$

Based on previous well-known research results on the vibration of second-order systems [6], [7], we consider two cases of k_1, k_2 (see Table 1).

Table 1 Candidates for k_1, k_2.

Case A	k_1=100, k_2=20
Case B	k_1=50, k_2=$10\sqrt{2}$

In this study, we desire our shifting controller to provide the most rapid response according to the shifting command without overshoot; we designate the settling time of the shifting system (the time in reaching the new equilibrium state) to be less than 1 *second*. Hence, we select the system damping ratio ζ to 1, which corresponds to the case of *critical damping*. For a given initial excitation, a critically damped system tends to approach the equilibrium position the fastest without any overshoot. Moreover, these feedback gains guarantee the asymptotic stability and tracking performance of the S-CVT shifting system.

5.4 Numerical Results

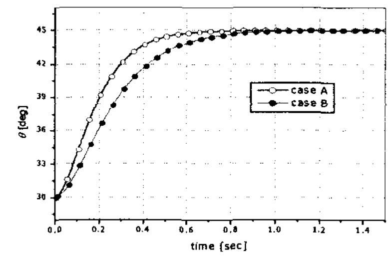
We first investigate the stability of the shifting system with the proposed feedback gains. To do this, we simulate the behaviors of the shifting system numerically.

For the simulation conditions, we set the initial states of the system to

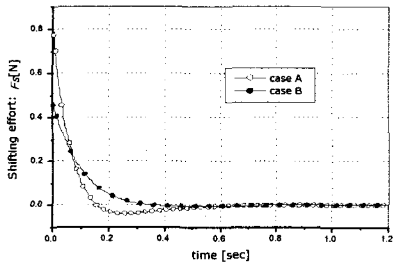
$$\theta=30^\circ, \dot{\theta}=0, F_i=1, F_o=\sqrt{3}\,.$$

This initial condition is one of the equilibrium points of the shifting system. At this instant, however, the output force suddenly changes from $\sqrt{3}$ to 1. Thus the input-output force equilibrium no longer holds, and the gear ratio (*i.e.*, the variator angle θ) of the S-CVT must be changed into a new equilibrium state which makes the system stable. Figure 4 shows the numerical results of the system behavior and corresponding control from Equation (9).

As expected, both cases of feedback gains show the asymptotic stability of the system. The variator angles for each case change from the initial state into the new equilibrium point $\theta=45^\circ$.



(a) Variator angle.



(b) Control.

Fig. 4 Stability of the shifting controller.

Next we investigate the tracking performance of the shifting system as follows. For the reference trajectory of the variator angle, we consider a sinusoidal function

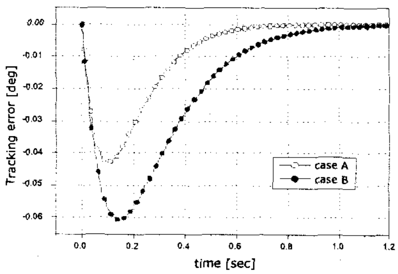
$$\theta_d=\frac{\pi}{3}\sin(\frac{\pi}{2}t-\phi)\,,$$

with the initial states of the system chosen as

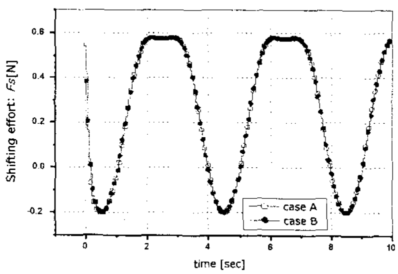
$$\theta=45^\circ, \dot{\theta}=0, F_i=1, F_o=1$$

Maintaining the input-output force as the initial values, the variator angle changes are calculated using Equation (10).

As expected, both cases of feedback gains show asymptotic convergence of tracking error. The relevant tracking error and corresponding shifting effort are shown in Figure 5.



(a) Tracking error.



(b) Control.

Fig. 5 Tracking error and corresponding control.

For both cases, the system responses match our predefined performance measure. From the numerical results, we select the feedback gains for case B, although the shifting response for case A is faster than that for case B (the time to reach the new equilibrium variator angle 45° in case A is almost 0.7 *second*, while for case B it is almost 1.0 *second*). The shifting effort (*i.e.*, control effort) for case B maintains a small value and varies monotonically compared to case A.

Using the selected feedback gains, we reconsider the stability of the system. The overall system behavior during a gear ratio change is determined from the S-CVT dynamics (2). The rotational speeds of the input, output, and sphere are depicted in Figure 6, using the initial condition and corresponding control in stability analysis.

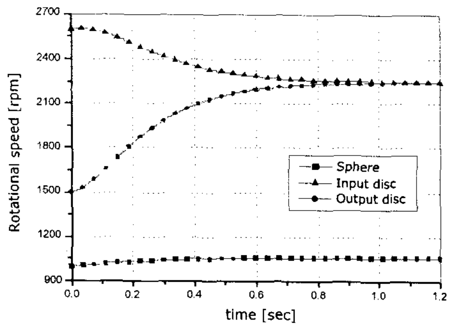


Fig. 6 System behaviors of S-CVT during shifting.

6. Conclusion

Due to the nonlinearity of the S-CVT shifting dynamics, the original open-loop system is inherently unstable. Hence a feedback controller is necessary to make the system stable and to achieve effective tracking performance. To do this, we designed a feedback controller that cancels nonlinearities and transforms the original nonlinear system dynamics into a stable and controllable linear one, based on the input-state linearization method.

In this paper, we showed the instability of the original S-CVT shifting system using Lyapunov's linearization method. With the differential geometric background, we performed the input-state linearization of our system, and designed a feedback controller which achieves asymptotic stability and effective tracking performance of the S-CVT shifting system. In selecting the feedback gains of the proposed controller, we considered our linearized shifting dynamics as a canonical second-order oscillation problem. In order to achieve a predefined shifting performance, we then set the feedback gains; comparing the numerical results of the shifting effort (*i.e.*, control effort) and the settling time. Finally we presented numerical results that demonstrate shifting controller performance with respect to stability and tracking.

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