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Natural Frequencies for Inhomogeneous Beams by Differential Transformation

미분변환에 의한 비균질 보의 진동해석

Kwon-Kyung Mun*, Shin-Yung Jae and Yung-Soon Ryu*****
권경문·신영재·유영순

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ABSTRACT

This paper presents the application of the technique of differential transformation to find the vibration frequencies for inhomogeneous beams with one sliding support , the other clamped and the other pinned boundary conditions. Numerical calculations are carried out. The frequencies obtained from the differential-transformation solutions are compared to published results to demonstrate the accuracy and flexibility of the method.

요 약

본 연구는 비균질 보의 진동해석에 새로운 선형 및 비선형 비분방정식의 해석인 미분변환방법을 적용하였으며, 미끄럼지지와 고정지지 및 미끄럼지지와 핀 지지의 경계조건을 고려하여 비균질 보에 대한 수치해석을 수행하였다. 본 해석법의 타당성을 검증하기 위하여 기존의 연구결과와 비교 검토하였으며, 그 결과 본 연구에 의한 해석결과가 기존의 것과 잘 일치함을 알 수 있었다.

1. Introduction

The problem of determining the natural frequencies and mode shapes of inhomogeneous beams is of importance in the design of many engineering components. The inhomogenity of a beam could arise from variations of its mass per unit area which, in turn, could be due to variations in the thickness of the beam or variations of the beam's material density.

Elishakoff and Rollot investigated the buckling of a variable stiffness column[1]. The closed form solutions for natural frequency of inhomogeneous beam with four boundary conditions (pinned-pinned, pinned-clamped, clamped-free and clamped-clamped) were given by Elishakoff and Candan[2]. Elishakoff and Becquet have obtained the closed form solutions for natural frequency of inhomogeneous beams under other boundary conditions. In these cases, polynomial representation of the mode shape was postulated, and a closed-form solution was obtained by formulating an inverse vibration problem[3,4].

In this study, differential transformation method is applied to analyze the vibration frequencies of inhomogeneous beams with one sliding support, the other clamped and the other pinned boundary conditions. The concept of this transformation was first proposed by Zhou in 1986 and was applied to solve the liner and nonlinear initial value problems in electric circuit[5].

Numerical calculations are carried out. The frequencies obtained from the differential-transformation solutions are compared to published results to demonstrate the accuracy and flexibility of the method.

2. Governing Differential Equation

The equation of motion governing the bending vibration of inhomogeneous beams is given as follows:

$$\frac{\partial^2}{\partial x^2}\left[\bar{E}(x)I\frac{\partial^2 w(x)}{\partial x^2}\right] = -\bar{\rho}(x)A\frac{\partial^2 w(x)}{\partial t^2} \qquad (1)$$

where $\bar{E}(x)$ is the Young's Modulus of the beam, I is the moment of inertia, $w(x,t)$ is the lateral displacement, $\bar{\rho}(x)$ is the material density, and A is the cross-sectional area. The right term of equation(1) represents the inertia forces during vibrations.

Assuming a steady-state solution

$$w(x,t) = w(x)e^{i\omega t} \qquad (2)$$

Substituting equation(2) into equation(1), we can obtain the ordinary differential equation of mode shape as follows

* 안동대학교 기계공학부 대학원
E-mail : munkk@pyunji.andong.ac.kr
Tel : (054)820-5077, Fax : (054) 823-5495

** 안동대학교 기계공학부

*** 안동대학교 기계공학부

$$\frac{d^2}{dx^2}\left[\overline{E}(x)\frac{d^2w(x)}{dx^2}\right]-\frac{\overline{\rho}(x)}{I}A\omega^2 w=0 \qquad (3)$$

Introducing the new variable $\xi=\frac{x}{l}$ in equation (3), we can obtain the governing equation for the displacement $w(\xi)$

$$\frac{d^2}{d\xi^2}\left[\overline{E}(\xi)\frac{d^2w(\xi)}{d\xi^2}\right]-\frac{\omega^2AL^4}{I}\overline{\rho}(\xi)w(\xi)=0 \qquad (4)$$

where, $\overline{E}(\xi)$, $\overline{\rho}(\xi)$ are the non dimensional material properties of the beam and ω is the natural frequency of the beam.

We assume that the material properties are

$$\overline{\rho}(\xi)=\rho_0\sum_{i=0}^{p}\overline{a_i}\xi^i \qquad (5)$$

$$\overline{E}(\xi)=E_0\sum_{i=0}^{q}\overline{b_i}\xi^i \qquad (6)$$

p and q are the degrees of the polynomial functions $\overline{\rho}(\xi)$ and $\overline{E}(\xi)$, respectively. The governing equation for a inhomogeneous beam is a 4th order ordinary differential equation. For a well-posed problem, it requires four boundary conditions. Theses can be obtained by imposing two boundary conditions at the end $\xi=0$, and another two boundary conditions at the end $\xi=1$. We consider the two different sets of a clamped - sliding support and sliding support - pinned boundary conditions.

If a beam in transverse vibration is clamped, the bending moment and shear force are not restricted, but the deflection and slope must vanish at that end:

$$w=0,\ w'=0,$$

at a sliding supported end, the slope or rotation is zero and no shear force is allowed:

$$w'=0,\ w'''=0$$

at a pinned end, the slope and shear force are unrestricted and the deflection and bending moment must vanish:

$$w=0,\ w''=0$$

3. Differential Transformation

Let $y(x)$ be an analytic in domain D and $x=0$ be a point in D. Then there exists precisely one power series with center at $x=0$ which represents $y(x)$; this series, the Maclaurin series of the function $y(x)$, is of form

$$y(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^ky(x)}{dx^k}\right]_{x=0} \quad \text{for } \forall x\subset D \qquad (7)$$

If we define differential transformation of function $y(x)$ as follows

$$Y(k)=\frac{1}{k!}\left[\frac{d^ky(x)}{dx^k}\right]_{x=0} \qquad (8)$$

and substitute equation (13) into equation (12), equation (12) becomes

$$y(x)=\sum_{k=0}^{\infty}x^kY(k) \qquad (9)$$

$Y(k)$ is the differential transformation (T-function) for the original function $y(x)$, and equation(8) is the differential inverse transformation of $Y(k)$ (5,6,7,8).

From the above definition of the differential transformation of the function, we can derive the rules of transformational operations: some of these, which are useful in the following analysis, are as presented in Table 1.

Table 1 Examples of the differential transformation of the original function

Original function	T-function
$w(x)=y(x)\pm z(x)$	$W(k)=Y(k)\pm Z(k)$
$z(x)=\lambda y(x)$	$Z(k)=\lambda Y(k)$
$w(x)=\frac{d^ny(x)}{dx^n}$	$W(k)=(k+1)(k+2)\cdots(k+n)Y(k+n)$
$w(x)=y(x)z(x)$	$W(k)=\sum_{l=0}^{k}Y(l)Z(k-l)$
$w(x)=x^m$	$W(k)=\delta(k-m)$ at $1\ \ k=m$, $0\ \ k\neq m$

In actual applications, the function y(x) may be expressed by a finite series and equation(9) can be written as

$$y(x)=\sum_{k=0}^{n}x^kY(k) \qquad (10)$$

Equation(10) implies that $\sum_{k=n+1}^{\infty}x^kY(k)$ is neglected. Generally, n is decided by the desired convergence of the natural frequency.

4. Application of Differential Transformation to Inhomogeneous Beams

Taking differential transformation of equation(4) and using the transformational operations mentioned above, we obtain

$$\sum_{l=0}^{k} b(l)(k-l+1)(k-l+2)(k-l+3)(k-l+4)W(k-l+4)$$
$$+2\sum_{l=0}^{k}(l+1)b(l+1)(k-l+1)(k-l+2)(k-l+3)W(k-l+3)$$
$$+\sum_{l=0}^{k}(l+1)(l+2)b(l+2)(k-l+1)(k-l+2)W(k-l+2)$$
$$-\frac{\omega^2 AL^4}{E_0 I}\rho_0\sum_{l=0}^{k}a(l)W(k-l)=0 \qquad (11)$$

where $W(k)$, $b(k)$, and $a(k)$ are T-functions of $w(\xi)$, $E(\xi)$, and $\rho(\xi)$, respectively.

In order to obtain the natural frequency of the inhomogeneous beams, the boundary conditions should be transformed. The above mentioned boundary conditions at each end should be obtained by differential transformation method as follows:

At the end $\xi = 0$

Clamped end :

$$W(0)=0 \text{ and } W(1)=0 \qquad (12)$$

Sliding support end:

$$W(1)=0 \text{ and } W(3)=0 \qquad (13)$$

At the end $\xi = 1$

Sliding support end:

$$\sum_{n=0}^{k} L^{n-1} n \xi^{n-1} W(n)=0 \quad \text{and}$$
$$\sum_{n=0}^{k} L^{n-3} n(n-1)(n-2)\xi^{n-3}W(n)=0 \qquad (14)$$

Pinned end :

$$\sum_{n=0}^{k} L^{n}\xi^{n}W(n)=0 \quad \text{and}$$
$$\sum_{n=0}^{k} L^{n-2} n(n-1)\xi^{n-2}W(n)=0 \qquad (15)$$

5. Numerical Analysis and Discussions

In order to obtain the natural frequencies of an inhomogeneous beam, we should take advantage of the transformed equation(11) and the four corresponding boundary conditions among equations(12-15). These can be represented in matrix form as

$$\begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} & c_{1,n+1} \\ c_{2,1} & c_{2,2} & \vdots & c_{2,n} & c_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n+1,1} & c_{n+1,2} & \cdots & c_{n+1,n} & c_{n+1,n+1} \end{bmatrix} \begin{bmatrix} W(0) \\ W(1) \\ \vdots \\ W(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (16)$$

A non-trivial solution exists when the determinant of the coefficient matrix vanishes. This condition leads to the following frequency equation:

$$\begin{vmatrix} c_{1,1} & c_{1,2} & \cdots & \cdots & \cdots & c_{1,n} & c_{1,n+1} \\ c_{2,1} & c_{2,2} & \cdots & \cdots & \cdots & c_{2,n} & c_{2,n+1} \\ \cdot & \cdot & \cdots & \cdots & \cdots & \cdot & \cdot \\ c_{n+1,1} & c_{n+1,2} & \cdots & \cdots & \cdots & c_{n+1,n} & c_{n+1,n+1} \end{vmatrix} = 0 \qquad (17)$$

The solution of equation(16) yields the desired frequency parameter $\omega_i^{(n)}$ of an inhomogeneous beam.

Where, $\omega = \omega_i^{(n)}$, $i = 1,2,...$ (18)

$\omega_i^{(n)}$ is the i-th estimated natural frequency corresponding to n, with n being decided by the following equation.

$$\left|\omega_i^{(n)} - \omega_i^{(n-1)}\right| \le \varepsilon \qquad (19)$$

where $\omega_i^{(n-1)}$ is the i-th estimated natural frequency corresponding to $n-1$ and ε is a tolerance parameter.

Substituting ω obtained above into equation(16), solving $W(0), W(1), W(2),...W(n)$ and substituting these into equation (17), we obtain the i-th mode shape of an inhomogeneous beams.

To validate the method introduced in this study and compare the calculated result with the previous work, two boundary conditions of inhomogeneous beams were considered.

Case I:

Inhomogeneous beam with clamped - sliding support boundary condition.

In this case, the transformed boundary condition equations become equations(12) and (15). The beam properties are given in Table 2.

Table 2 Material properties of rectangular beam

Young's Modulus (E_0)	$2.068 \times 10^{11} N/m^2$
Density (ρ_0)	$7850 kg/m^3$
Width (b)	$0.25m$
Height (h)	$0.25m$
Length (L)	$10m$

we consider two specific cases, which are associated with constant and linear variations of the density[3].

Table 3 The fundamental natural frequency for inhomogeneous beam with clamped - sliding supported boundary condition. (The fundamental natural frequency of homogeneous beam: 20.71856 (Hz))

Density variations	Young's modulus	Fundamental natural frequency	Reference[4]
Constant	$E(\xi) = E_0(1 + 0.54545\xi + 0.13636\xi^2 - 0.40909\xi^3 + 0.010227\xi^4)$	22.47610 *(Hz)*	22.47610 *(Hz)*
Linear	$E(\xi) = E_0(1 + 0.5829\xi + 0.24874\xi^2 - 0.12814\xi^3 - 0.1639\xi^4 + 0.056633\xi^5)$	13.98105 *(Hz)*	13.98105 *(Hz)*

Table 4 The fundamental natural frequency for inhomogeneous beam with sliding support - pinned boundary condition. (The fundamental natural frequency of homogeneous beam: 9.13965 (Hz)

Density variations	Young's modulus	Fundamental natural frequency	Reference [3]
Constant	$E(\xi) = E_0(1 - 0.229508\xi^2 + 0.0163934\xi^4)$	8.99863 (Hz)	8.99863 (Hz)
Linear	$E(\xi) = E_0(1 + 0.274005\xi - 0.229508\xi^2 - 0.135861\xi^3 + 0.0163934\xi^4 + 0.0.01171\xi^5)$	8.99863 (Hz)	8.99863 (Hz)
Parabolically	$E(\xi) = E_0(1 + 0.24567\xi - 0.102362\xi^2 - 0.121785\xi^3 - 0.065617\xi^4 + 0.01049858\xi^5 + 0.00787401\xi^6)$	8.52065 (Hz)	8.52065 (Hz)
Cubic polynomial	$E(\xi) = E_0(1 + 0.299038\xi - 0.102362\xi^2 - 0.0684164\xi^3 - 0.06561\xi^4 - 0.046369\xi^5 + 0.0078740\xi^6 + 0.00612423\xi^7)$	8.52065 (Hz)	8.52065 (Hz)
Quartic polynomial	$E(\xi) = E_0(1 + 0.2900051\xi - 0.069065\xi^2 - 0.06635\xi^3 - 0.0334295\xi^4 - 0.0449686\xi^5 - 0.0334295\xi^6 + 0.00593924\xi^7 + 0.0047514\xi^8)$	8.39098 (Hz)	8.39098 (Hz)

1) constant density

$\rho(\xi) = \rho_0$,

$$E(\xi) = E_0(1 + 0.54545\xi + 0.13636\xi^2 - 0.40909\xi^3 + 0.010227\xi^4)$$

2) linearly varying density

$\rho(\xi) = \rho_0(1 + 2\xi)$,

$$E(\xi) = E_0(1 + 0.5829\xi + 0.24874\xi^2 - 0.12814\xi^3 - 0.16394\xi^4 + 0.056533\xi^5)$$

In order to find the fundamental natural frequency for inhomogeneous beam with constant density variation, We calculate up to the 8th the terms $W(8)$, where n=8 is decided by the convergence of the natural frequency.

From equation(17) , we have the frequency equation as follow:

$$W_1^8 = 7.889596 \times 10^{52} + 3.196582 \times 10^{50}\omega^2 + 4.295502 \times 10^{47}\omega^4 = 0 \qquad (20)$$

Solving equation(20), we have the first natural frequency:

$$\omega_1^{(8)} = 22.47610 \qquad (21)$$

When n=7, by the same way, we obtain

$$\omega_1^{(7)} = 22.47610 \qquad (22)$$

From equation (21) and (22) we have

$$\left|\omega_1^{(8)} - \omega_1^{(7)}\right| = 0 \qquad (23)$$

Moreover, we obtain the fundamental natural frequency of inhomogeneous beam with clamped-sliding supported boundary conditions.

Table 3 shows that the calculated results are very much the same as those obtained by Elishakoff [4].

Case II:

Inhomogeneous beam with sliding support - pinned boundary conditions. The boundary condition equations are equation (13) and (14).

The beam properties are the same as the above mentioned case.

In this case, we treat five specific cases, which are associated with constant, linear, parabolic, cubic and quartic variations of the density [3].

1) constant density

$\rho(\xi) = \rho_0$,

$$E(\xi) = E_0(1 - 0.229508\xi^2 + 0.0163934\xi^4)$$

2) linearly varying density

$\rho(\xi) = \rho_0(1+\xi)$,

$E(\xi) = E_0(1 + 0.274005\xi - 0.229508\xi^2 - 0.135831\xi^3 + 0.0163934\xi^4 + 0.01171\xi^5)$

3) parabolically varying density

$\rho(\xi) = \rho_0(1+\xi+\xi^2)$,

$E(\xi) = E_0(1 + 0.245675\xi - 0.102362\xi^2 - 0.121785\xi^3 - 0.062617\xi^4 + 0.01049858\xi^5 + 0.00787401\xi^6)$

4) density as a cubic polynomial

$\rho(\xi) = \rho_0(1+\xi+\xi^2+\xi^3)$,

$E(\xi) = E_0(1 + 0.299038\xi - 0.102362\xi^2 - 0.0684164\xi^3 - 0.065617\xi^4 - 0.04463692\xi^5 + 0.00787401\xi^6 + 0.00612423\xi^7)$

5) *density as a quartic polynomial*

$\rho(\xi) = \rho_0(1+\xi+\xi^2+\xi^3+\xi^4)$,

$E(\xi) = E_0(1 + 0.2900051\xi - 0.069065\xi^2 - 0.06635\xi^3 - 0.033429\xi^4 - 0.0449686\xi^5 - 0.0334295\xi^6 + 0.00593924\xi^7 + 0.0047514\xi^8)$

Table 4 shows that the calculated results are very much the same as those obtained by Elishakoff [3].

6. Conclusions

The vibration frequencies of inhomogeneous beams with one sliding support, the other clamped and the other pinned boundary conditions were analyzed by using Differential Transformation. The obtained results were compared to the published works to demonstrate the accuracy and flexibility of the method.

The present analysis shows the usefulness and validity of differential transformation in solving the vibration frequencies for inhomogeneous beams.

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