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GLOBAL BIFURCATION ANALYSIS OF NON-LINEAR OSCILLATION OF A RECTANGULAR PLATE

직사각형 평판의 비선형 진동의 광역분기해석

SEO IL CHANG, JAE-WON LEE, JIN-KWEN CHOI
장 서일, 이 재원, 최 진권

Department of Environmental Engineering
The University of Seoul
90 Jeonnong-dong, Dongdaemun-gu, Seoul 130-743, Korea
schang@uos.ac.kr

Abstract

직사각형 평판이 수직방향으로 조화가진력을 받을 때 그 변위가 큰 경우 두 개의 모드 간의 비선형적 상호작용에 대한 연구이다. 폰 칼만 운동방정식에서 유도된 두 개의 상미분 방정식으로부터 수차에 걸친 좌표변환을 거쳐 자유진동의 경우 정지해와 주기해를 구한다. 말굽형태의 분기현상이 일어날 수 있는 조건을 호모클리닉 또는 헤테로클리닉 궤적의 유무로부터 결정한다. 혼돈현상의 발생조건을 구하기 위해 멜니코프 방법이 적용되어질 수 있는 형태로 변환하여 광역섭동법의 수학적 결과를 직접적으로 적용할 수 있는 형태로 변환한다.

INTRODUCTION

In this paper, global dynamics in an externally excited thin rectangular plate is studied. For appropriate aspect ratio, two or more modes of a rectangular plate have identical natural frequencies. Through Galerkin's procedure, the von Karman plate equations can be reduced to two coupled non-linear ordinary differential equations of the two modes [1]. The method of averaging is used to transform the modal equations to four first-order differential equations representing the slow-time evolution of amplitudes and phases of harmonic motions of the two modes. The system of equations is transformed by a sequence of canonical transformations to the system of appropriate form, to which Melnikov method can be applied to study global dynamics of the system by adopting the result of Kovacic and Wiggins [2].

By following the ideas in Holmes [3] and Feng and Sethna [4], open sets in parameter space are identified to find phase portraits of, the so called, "unperturbed system". The geometric structures of the unperturbed system are studied to find orbits homoclinic and heteroclinic to fixed points. It is known that under certain condition, these orbits break and cause the onset of chaotic motion for the so called, "perturbed system".

FORMULATION OF THE PROBLEM

Reduction procedures [1] by Galerkin's method is omitted here and the two coupled non-linear differential equations of the two modes in one-to-one internal resonance can be obtained as follows:

$$\ddot{X}_1 + X_1 = \epsilon\left[(A_1X_1^2 + A_2X_2^2)X_1 - c\dot{X}_1 + Q_1\cos\tau + \sigma_1X_1\right],$$
$$\ddot{X}_2 + X_2 = \epsilon\left[(A_2X_1^2 + A_3X_2^2)X_2 - c\dot{X}_2 + Q_2\cos\tau + \sigma_2X_2\right], \tag{1}$$

where X_1, X_2, Q_1, Q_2, c, σ_1, andσ_2 represent amplitudes of the two modes, amplitudes of external forces which excite the first and second modes directly, damping constant and frequency detuning parameters, respectively. A_1, A_2, andA_3 are the coefficients which can be determined by the selection of the two modes. ϵ is a small parameter.

The discretized equations (1) of motion for the two-mode approximation can be rewritten in state space form as:

$$\begin{aligned}
\dot{q}_1 &= \dot{X}_1 = +\frac{\partial H}{\partial p_1} = p_1\,,\\
\dot{p}_1 &= \ddot{X}_1 = -\frac{\partial H}{\partial q_1} - \epsilon c p_1\\
&= -q_1 + \epsilon\left[(A_1q_1^2 + A_2q_2^2)\,q_1 - cp_1 + Q_1\cos\tau + \sigma_1q_1\right],\\
\dot{q}_2 &= \dot{X}_2 = +\frac{\partial H}{\partial p_2} = p_2\,,\\
\dot{p}_2 &= \ddot{X}_2 = -\frac{\partial H}{\partial q_2} - \epsilon c p_2\\
&= -q_2 + \epsilon\left[(A_2q_1^2 + A_3q_2^2)\,q_2 - cp_2 + Q_2\cos\tau + \sigma_2q_2\right],
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
H &= \;= H_0 + \epsilon H_1 = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)\\
&- \epsilon\left[\frac{1}{4}(A_1q_1^4 + 2A_2q_1^2q_2^2 + A_3q_2^4) + (Q_1q_1 + Q_2q_2)\cos\tau + \frac{1}{2}(\sigma_1q_1^2 + \sigma_2q_2^2)\right].
\end{aligned} \tag{3}$$

Transforming equations (2) by using the canonical transformation in action and angle variables:

$$q_i = (2I_i)^{\frac{1}{2}}\sin(\theta_i + \tau)\,, \quad p_i = (2I_i)^{\frac{1}{2}}\cos(\theta_i + \tau)\,,\; i = 1,\,2, \tag{4}$$

and time-averaging the resulting equations, gives

$$\begin{aligned}
I_1' &= -\frac{\partial \bar{H}}{\partial \theta_1} - cI_1\\
&= -\frac{1}{2}A_2I_1I_2\sin 2(\theta_1 - \theta_2) + \frac{Q_1}{\sqrt{2}}\sqrt{I_1}\cos\theta_1 - cI_1\,,\\
\theta_1' &= \frac{\partial \bar{H}}{\partial I_1}\\
&= -\frac{1}{4}\left[A_2I_2\cos 2(\theta_1 - \theta_2) + 2A_2I_2 + 3A_1I_1 + 2\sigma_1\right] - \frac{\sqrt{2}}{4}Q_1\frac{\sin\theta_1}{\sqrt{I_1}}\,,
\end{aligned}$$

$$
\begin{aligned}
I_2' &= -\frac{\partial \bar{H}}{\partial \theta_2} - cI_2 \\
&= \frac{1}{2} A_2 I_1 I_2 \sin 2(\theta_1 - \theta_2) + \frac{1}{\sqrt{2}} Q_2 \sqrt{I_2} \cos \theta_2 - cI_2 \\
\theta_2' &= \frac{\partial \bar{H}}{\partial I_2} \\
&= -\frac{1}{4} [A_2 I_1 \cos 2(\theta_1 - \theta_2) + 2A_2 I_1 + 3A_3 I_2 + 2\sigma_2] - \frac{\sqrt{2}}{4} Q_2 \frac{\sin \theta_2}{\sqrt{I_2}} ,
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\bar{H} &= \bar{H}_0 + \bar{H}_1, \\
\bar{H}_0(I, \theta) &= -\frac{1}{8} [3A_1 I_1^2 + 4A_2 I_1 I_2 + 3A_3 I_2^2 \\
&\quad + 4 (\sigma_1 I_1 + \sigma_2 I_2) + 2A_2 I_1 I_2 \cos 2(\theta_1 - \theta_2)] , \\
\bar{H}_1(I, \theta) &= -\frac{1}{\sqrt{2}} \left[Q_1 \sqrt{I_1} \sin \theta_1 + Q_2 \sqrt{I_2} \sin \theta_2 \right] ,
\end{aligned} \tag{6}
$$

(7)

where the $'$ notation stands for a derivative with respect to slow time $\epsilon\tau$. Finally, transforming from the action-angle variables (I_i, θ_i) to new variables (P_i, Θ_i) with a generating function $F = (\theta_1 - \theta_2)P_1 + \theta_2 P_2$ via the coordinate changes:

$$
I_1 = P_1, \ \theta_1 = \Theta_1 + \Theta_2, \ I_2 = P_2 - P_1 \text{ and } \theta_2 = \Theta_2, \tag{8}
$$

yields,

$$
\begin{aligned}
P_1' &= -\frac{\partial K}{\partial \Theta_1} - cP_1 \\
&= \frac{1}{2} A_2 P_1 (P_1 - P_2) \sin 2\Theta_1 + \frac{Q_1}{\sqrt{2}} \sqrt{P_1} \cos (\Theta_1 + \Theta_2) - cP_1 , \\
\Theta_1' &= + \frac{\partial K}{\partial P_1} \\
&= \frac{1}{4} [A_2(2P_1 - P_2)(2 + \cos 2\Theta_1) + 3A_3(P_2 - P_1) - 3A_1 P_1 + 2(\sigma_2 - \sigma_1)] \\
&\quad + \frac{\sqrt{2}}{4} \left[-Q_1 \frac{\sin (\Theta_1 + \Theta_2)}{\sqrt{P_1}} + Q_2 \frac{\sin \Theta_2}{\sqrt{P_2 - P_1}} \right] , \\
P_2' &= -\frac{\partial K}{\partial Q_2} - cP_2 \\
&= \frac{1}{\sqrt{2}} \left[Q_1 \sqrt{P_1} \cos (\Theta_1 + \Theta_2) + Q_2 \sqrt{P_2 - P_1} \cos \Theta_2 \right] - cP_2 , \\
\Theta_2' &= + \frac{\partial K}{\partial P_2} \\
&= \frac{1}{4} [- A_2 P_1 (2 + \cos 2\Theta_1) + 3A_3 (P_1 - P_2) - 2\sigma_2] - \frac{\sqrt{2}}{4} Q_2 \frac{\sin \Theta_2}{\sqrt{P_2 - P_1}} ,
\end{aligned} \tag{9}
$$

where

$$
K = K_0 + K_1,
$$

$$
\begin{aligned}
K_0(P,\Theta) &= -\frac{1}{8}\ [(\ 3A_1 - 4A_2 + 3A_3)\ P_1^2\ +\ 2(2A_2 - 3A_3)\ P_1\ P_2\ +\ 3A_3\ P_2^2 \\
&\quad + 4\ (\sigma_1\ -\ \sigma_2)\ P_1\ +\ 4\sigma_2\ P_2\ +\ 2A_2 P_1 \cos 2\Theta_1\ (P_2 - P_1)]\ , \\
K_1(P,\Theta) &= -\frac{1}{\sqrt{2}}\left[\ Q_1\ \sqrt{P_1}\ \sin\ (\Theta_1 + \Theta_2)\ +\ Q_2\sqrt{P_2 - P_1}\ \sin\ \Theta_2\ \right].
\end{aligned}
\tag{10}
$$

Bifurcations of Unperturbed System and Heteroclinic Orbits

We consider now the unperturbed system, that is, a system of free vibration without any external excitation force or any damping. From equations (9), we get the equations of the unperturbed system as follows:

$$
\begin{aligned}
P_1' &= -\frac{\partial K_0}{\partial Q_1} = \frac{1}{2}\ A_2 P_1 (P_1\ -\ P_2)\ \sin\ 2\Theta_1\ , \\
\Theta_1' &= +\frac{\partial K_0}{\partial P_1} \\
&= \frac{1}{4}[A_2(2P_1 - P_2)(2 + \cos\ 2\Theta_1) + 3A_3(P_2 - P_1) - 3A_1 P_1 + 2(\sigma_2 - \sigma_1)]\ , \\
P_2' &= -\frac{\partial K_0}{\partial \Theta_2} = 0\ , \\
\Theta_2' &= +\frac{\partial K_0}{\partial P_2} = \frac{1}{4}[\ -\ A_2 P_1(2\ +\ \cos\ 2\Theta_1)\ +\ 3A_3(P_1\ -\ P_2)\ -\ 2\sigma_2]\ .
\end{aligned}
\tag{11}
$$

Since in equation (11) $P_2' = 0$, P_2 remains constant and we need only study the (P_1, Θ_1)-system. After integration of this system for fixed P_2, $\Theta_2(t)$ can be solved for directly. We note again that equations (11) form a completely integrable Hamiltonian system with $P_2 = P_{20}$, a constant. We first study this reduced one degree-of-freedom system:

$$
\begin{aligned}
P_1' &= -\frac{\partial K_0^0}{\partial \Theta_1} = \frac{1}{2}\ A_2 P_1 (P_1\ -\ P_{20})\ \sin\ 2\Theta_1\ , \\
\Theta_1' &= +\frac{\partial K_0^0}{\partial P_1} \\
&= \frac{1}{4}[A_2(2P_1 - P_{20})(2 + \cos\ 2\Theta_1) + 3A_3(P_{20} - P_1) - 3A_1 P_1 + 2(\sigma_2 - \sigma_1)]\ ,
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
K_0^0 &= (\ K_0\ )_{P_2 = P_{20}} \\
&= -\frac{1}{8}\ [(\ 3A_1 - 4A_2 + 3A_3)\ P_1^2\ +\ 2(2A_2 - 3A_3)\ P_1\ P_{20}\ +\ 3A_3\ P_{20}^2 \\
&\quad + 4\ (\sigma_1\ -\ \sigma_2)\ P_1\ +\ 4\sigma_2\ P_{20}\ +\ 2A_2 P_1 \cos 2\Theta_1\ (P_{20} - P_1)]
\end{aligned}
\tag{13}
$$

We note by inverting the transformations that

$$
P_2 = \frac{1}{2}\ [(q_1^2\ +\ q_2^2)\ +\ (p_1^2\ +\ p_2^2)]\ =\ H_0(p,q)
\tag{14}
$$

where $H_0(p,q)$, first defined in equation (3), is the total energy for a linear conservative system. Since $P_1 = I_1 \geq 0$ and $P_2 - P_1 = I_2 \geq 0$, $P_1 \geq 0$ and $P_2 \geq P_1$. Furthermore, since Θ_1 is periodic, we need to consider only $0 \leq P_1 \leq P_{20}$ and $0 \leq \Theta_1 \leq \pi$. The present study, however, includes more general pictures of phase flows and equilibrium points.

Consider now the equilibrium points of equations (12), that are defined by the solutions of $P_1' = 0$ and $\Theta_1' = 0$. To study the nature of these equilibrium points, the determinant of the Jacobian of equations (12) is evaluated at the equilibrium points. The Jacobian of (12) is

$$Jac = \begin{bmatrix} \frac{1}{2}A_2\ (2P_1 - P_{20})\sin 2\Theta_1 & A_2\ P_1\ (P_1 - P_{20})\cos 2\Theta_1 \\ \frac{1}{4}\{\ 2A_2\,(2+\cos 2\Theta_1) - 3A_3 - 3A_1\} & -\frac{1}{2}A_2\ (2P_1 - P_{20})\sin 2\Theta_1 \end{bmatrix} \tag{15}$$

and its determinant is

$$\begin{aligned} \det\ Jac &= -\frac{1}{4}A_2{}^2\ (2P_1 - P_{20})^2 \sin^2 2\Theta_1 \\ & \quad -\frac{1}{4}A_2 P_1\,(P_1 - P_{20})\ \cos\ 2\Theta_1\,[2A_2(2+\cos\ 2\Theta_1) - 3A_3 - 3A_1] \end{aligned} \tag{16}$$

Note that tr $Jac = 0$, since the system is Hamiltonian and that sign of the det Jac determines the stability of the equilibrium points. They are saddles when det $Jac < 0$, and centers when det $Jac > 0$. The four equilibrium points of equations (12) can be easily shown to be:

$$\begin{aligned} (i) &\quad P_1 = 0;\ \cos\ 2\Theta_1 = -2 + 3\,\frac{A_3}{A_2} + \frac{2}{A_2 P_{20}}\,(\sigma_2 - \sigma_1)\,, \\ (ii) &\quad P_1 = P_{20}\,;\ \cos\ 2\Theta_1 = -2 + 3\,\frac{A_1}{A_2} - \frac{2}{A_2 P_{20}}\,(\sigma_2 - \sigma_1)\,, \\ (iii) &\quad \Theta_1 = 0, \pi;\ P_1 = \frac{3(A_3 - A_2)}{-6A_2 + 3A_3 + 3A_1}\,P_{20} + \frac{2}{-6A_2 + 3A_3 + 3A_1}\,(\sigma_2 - \sigma_1)\,, \\ (iv) &\quad \Theta_1 = \frac{\pi}{2};\ P_1 = \frac{(3A_3 - A_2)}{-2A_2 + 3A_3 + 3A_1}\,P_{20} + \frac{2}{-2A_2 + 3A_3 + 3A_1}\,(\sigma_2 - \sigma_1)\,. \end{aligned} \tag{17}$$

From equations (17), we can see that the equilibrium points (i) and (ii) only exist if cos $2\Theta_1$ has values between ± 1. The determinant can be shown to be always negative in these cases and so the equilibrium points (i) and (ii) are always saddles. The equilibrium points (iii) and (iv) can be both saddles and centers depending on the sign of det Jac. Using these results it is possible to identify open sets in the $\sigma_2/P_{20} - \sigma_1/P_{20}$ plane where only certain types of equilibrium points are possible. The bifurcation sets separating the appropriate regions in parameter plane are shown in Figure 1.

The equilibrium points (i) are saddles if:

$$\frac{A_2 - 3A_3}{2} \geq \frac{\sigma_2 - \sigma_1}{P_{20}} \geq \frac{3(A_2 - A_3)}{2}. \tag{18}$$

This defines, respectively, the lines labelled 4 and 3 in Figure 1. Similarly, the equilibrium points (ii) are saddles if:

$$-\frac{A_2 - 3A_1}{2} \leq \frac{\sigma_2 - \sigma_1}{P_{20}} \leq -\frac{3(A_2 - A_1)}{2}. \tag{19}$$

This defines, respectively, lines 1 and 3 in Figure 1. The equilibrium points (iii) are saddles if:

$$\frac{3(A_2 - A_3)}{2} > \frac{\sigma_2 - \sigma_1}{P_{20}} > \frac{3(A_1 - A_2)}{2}, \tag{20}$$

表 1: Types of equilibrium points in the bifurcation sets

	(i)	(ii)	(iii)	(iv)
I	×	×	center	saddle
II	×	saddle	center	center
III	×	×	saddle	center
IV	saddle	×	center	center
V	×	×	center	saddle

and centers if this inequality is not satisfied. This also defines, respectively, lines 3 and 2 in Figure 1. The equilibrium points (iv) are saddles if:

$$\frac{\sigma_2 - \sigma_1}{P_{20}} > \frac{A_2 - 3A_3}{2}, \quad \text{or} \quad \frac{\sigma_2 - \sigma_1}{P_{20}} < \frac{-A_2 + 3A_1}{2}. \tag{21}$$

Otherwise these equilibrium points will be centers. These inequalities also define lines 4 and 1, respectively, in Figure 1. From Figure 1 we can see five clear regions. In Table 1 are summarized the results of this stability analysis. Using the values for A_1, A_2 and A_3 for (1,2) and (3,1) modes, the phase orbits for σ_1/P_{20} = .006, and σ_2/P_{20} = -.072 (region I), = -.03 (region II), = -.0012 (region III), = .03 (region IV), = .072 (region V), were calculated. These are shown in Figure 2. Note that only Figures 2 (b), (c) and (d) have heteroclinic orbits in the physically meaningful region, that is, when $0 \le P_1 \le P_{20}$. In Figure 2 (II) are shown the heteroclinic orbits connecting the saddle points (ii), in Figure 2 (III) are shown the heteroclinic orbits connecting the saddle points (iii), and in Figure 2 (IV) are shown the heteroclinic orbits connecting the saddle points (i).

Following Kovacic and Wiggins, we transform equations (9) by following transformation:

$$x = \sqrt{2P_1}\sin\Theta_1,\ y = \sqrt{2P_1}\cos\Theta_1,\ J = P_2,\ \theta = \Theta_2\ , \tag{22}$$

and then we get

$$\begin{aligned}
x' &= -\frac{3}{8}(A_1 - 2A_2 + A_3)y^3 - \frac{3}{8}(A_1 - \frac{4}{3}A_2 + A_3)yx^2 - \frac{3}{4}(A_2 - A_3)Jy - \frac{1}{2}(\sigma_1 - \sigma_2)y + g^x, \\
y' &= \frac{3}{8}(A_1 - \frac{2}{3}A_2 + A_3)x^3 + \frac{3}{8}(A_1 - \frac{4}{3}A_2 + A_3)y^2x + \frac{1}{4}(A_2 - 3A_3)Jx + \frac{1}{2}(\sigma_1 - \sigma_2)x + g^y, \\
\theta' &= -\frac{1}{8}(A_2 - 3A_3)x^2 - \frac{3}{8}(A_2 - A_3)y^2 - \frac{3}{4}A_3J - \frac{1}{2}\sigma_2 + g^\theta, \\
J' &= g^J,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
g^x &= -\frac{1}{2}Q_1\sin\theta + \frac{1}{2}\frac{Q_2\, y\sin\theta}{\sqrt{2J - x^2 - y^2}} - \frac{1}{2}c\, x, \\
g^y &= -\frac{1}{2}Q_1\cos\theta - \frac{1}{2}\frac{Q_2\, x\sin\theta}{\sqrt{2J - x^2 - y^2}} - \frac{1}{2}c\, y, \\
g^\theta &= -\frac{1}{2}\frac{Q_2\sin\theta}{\sqrt{2J - x^2 - y^2}}, \\
g^J &= \frac{1}{2}Q_1(y\cos\theta - x\sin\theta) + \frac{1}{2}Q_2\cos\theta\sqrt{2J - x^2 - y^2} - cJ,
\end{aligned} \tag{24}$$

Equations (23) have the general form

$$
\begin{aligned}
x' &= +\frac{\partial \bar{K}}{\partial y} - \frac{1}{2} c\, x, \quad y' = -\frac{\partial \bar{K}}{\partial x} \\
J' &= -\frac{\partial \bar{K}}{\partial \theta} - c\, J, \quad \theta' = +\frac{\partial \bar{K}}{\partial J},
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
\bar{K} &= \bar{K}_0 + \bar{K}_1, \\
\bar{K}_0 &= -1/2\,\sigma_2 J + \left(-\frac{3}{32} A_1 + 3/16\, A_2 - \frac{3}{32} A_3\right) y^4 + \left(-\frac{3}{32} A_3 - \frac{3}{32} A_1 + 1/16\, A_2\right) x^4 \\
&+ (-3/16\, A_3 - 3/16\, A_1 + 1/4\, A_2)\, y^2 x^2 + (-1/4\,\sigma_1 + 1/4\,\sigma_2)\, x^2 + (-1/4\,\sigma_1 + 1/4\,\sigma_2)\, y^2 \\
&+ (3/8\, A_3 - 3/8\, A_2)\, J y^2 + (3/8\, A_3 - 1/8\, A_2)\, x^2 J - 3/8\, A_3 J^2 \\
\bar{K}_1 &= -1/2\, Q_2 \sin(\theta) \sqrt{2\,J - x^2 - y^2} - 1/2\, Q_1 x \cos(\theta) - 1/2\, Q_1 y \sin(\theta).
\end{aligned}
\tag{26}
$$

Summary and Conclusion

Non-linear dynamics of an externally excited thin rectangular plate is studied by global perturbation technique. The method of averaging is used to transform the modal equations to four first-order differential equations representing the slow-time evolution of amplitudes and phases of harmonic motions of the two modes. The system of equations is transformed by a sequence of canonical transformations to the system of appropriate form, to which Melnikov method can be applied. Bifurcation sets in parameter space are identified to find phase portraits of unperturbed system. The geometric structures of the unperturbed system show orbits homoclinic and heteroclinic to fixed points.

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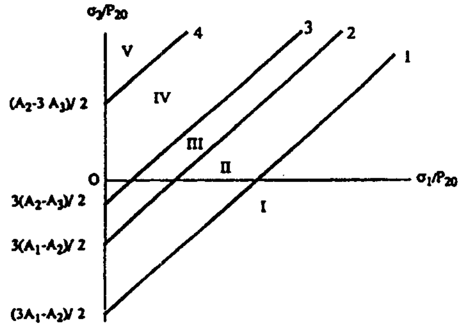

Figure 1. Bifurcation sets in $(\sigma_2/P_{20} - \sigma_1/P_{20})$ plane.

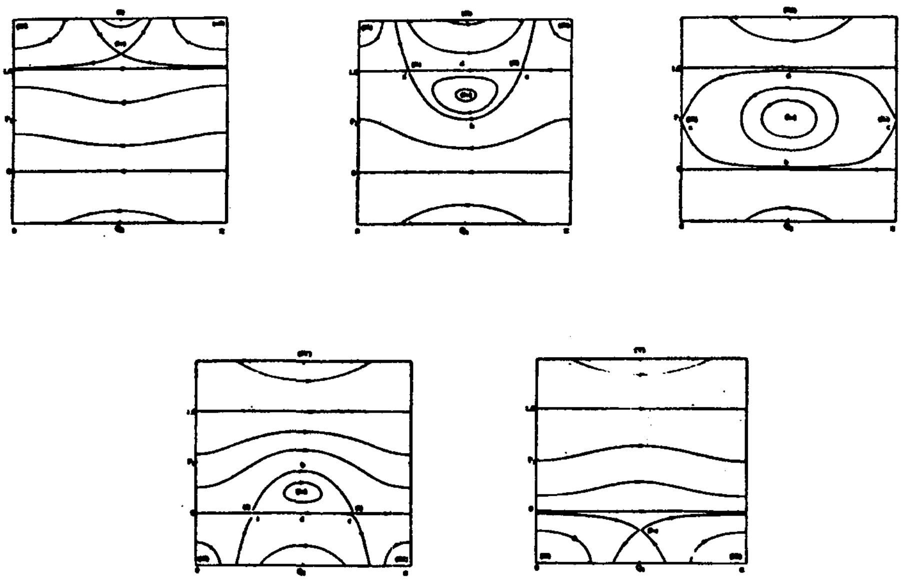

Figure 2. Solution orbits in (P_1, Q_1) phase plane for σ_1/P_{20} = .006, and σ_2/P_{20} = -.072 (I), = -.03 (II), -.0012 (III), = .03 (IV), = .072 (V).