전단변형을 고려한 곡선보의 미분구적법(DQM) 내평면 진동해석

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1. Introduction

The problem of the vibration of arches has become a subject of interest for many investigators due to its importance in many practical applications. The early investigators into the in-plane vibration of rings were Hoppe [1] and Love [2]. Love [2] improved on Hoppe's theory by allowing for stretching of the ring. Lamb [3] investigated the statics of incomplete ring with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog [4] used the Rayleigh-Ritz method for finding the lowest natural frequency of circular arcs with simply supported or clamped ends, and his work was extended by Volterra and Morell [5] for the vibrations of arches having center lines in the form of cycloids, catenaries, or parabolas. Archer [6] carried out for a mathematical study of the in-plane inextensional vibrations of an incomplete circular ring of small cross section with the basic equations of motion as given in Love [2] and gave a prescribed time - dependent displacement at the other end for the case of clamped ends. Nelson [7] applied the Rayleigh-Ritz method in conjunction with Lagrangian multipliers to the case of a circular ring segment having simply supported ends. Ojalvo [8] calculated the natural frequencies out-of-plane vibration of circular arches based on classical beam theory. Recently, Irie et al. [9] have analyzed circular arches based on Bresse-Timoshenko beam theory in which both rotatory inertia and shear deformation are taken into account.

A rather efficient alternate procedure for the solution of partial differential equations is the method of differential quadrature which was introduced by Bellman and Casti [10]. This simple direct technique can be applied to a large number of cases to circumvent the difficulties of programming complex algorithms for the computer, as well as excessive use of storage. This method is used in the present work to analyze the free in-plane shear deformable vibrations of curved beams. The lowest frequency parameters are calculated for the member of rectangular and circular cross section under clamped-clamped end conditions. Numerical results are compared with transfer matrix solution obtained by Irie et al. [9].

2. System and Governing Equations

The uniform curved beam considered is shown in Fig. 1. A point on the centroidal axis is defined by the angle θ, measured from the left support. a is the radius of the centroidal axis, and θ_0 is the opening angle. The radial and tangential displacements by u and w, respectively.

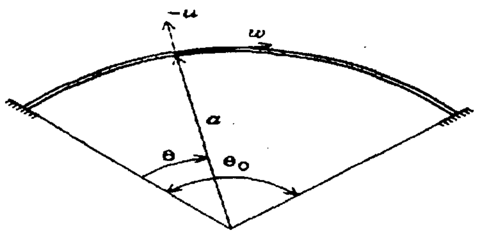

Fig. 1 Geometry of the circular arch for shear deformation

The differential equations governing the in-plane vibration of a circular arch based on the Bresse-Timoshenko beam theory, in which both rotatory inertia and shear deformation are taken into account were given by Irie et al. [9] as

$$\frac{k}{2(1+v)}\frac{\ddot{u}}{a\theta_0^2}-\left(1-\lambda^2\frac{1+k^2}{s_y^2}\right)\frac{u}{a}-\frac{k}{2(1+v)}\frac{\psi'}{\theta_0}+\left(1+\frac{k}{2(1+v)}\right)\frac{w'}{a\theta_0}=0, \quad (1)$$

$$\frac{k}{2(1+v)}\frac{u'}{a\theta_0}+k^2\frac{\psi''}{\theta_0^2}-\left(\frac{k}{2(1+v)}-\lambda^2\frac{k_2^2}{s_y^2}0\psi+\left(\frac{k}{2(1+v)}+\lambda^2\frac{k_1^2}{s_2^y}\right)\frac{w}{a}=0, \quad (2)$$

$$\left(1+\frac{k}{2(1+v)}\right)\frac{u'}{a\theta_0}-\left(\frac{k}{2(1+v)}+\lambda^2\frac{k_1^2}{s_y^2}\right)\psi-\frac{w''}{a\theta_0^2}+\left(\frac{k}{2(1+v)}-\lambda^2\frac{1+k^2}{s_y^2}\right)\frac{w}{a}=0, \quad (3)$$

in which each prime denotes one differentiation with respect to the dimensionless distance coordinate, X, defined as $X=\dfrac{\theta}{\theta_0}$

k is the shear correction factor depending on the shape of the cross section, v is the Poisson's ratio of the arch, and ψ is the slope of the displacement curve due to pure bending. For simplicity of the analysis, the following dimensionless variables have been introduced:

$$s_y^2 = A\,a^2/I_y \qquad \lambda^2 = \rho A\,a^4\,p^2/E\,I_y \tag{4}$$

where A is the cross-sectional area, I_y is the second area moment, ρ is the material density, p is the circular frequency, E is the Young's modulus of elasticity for the material, and s_y is the slenderness ratio of the arch. The quantities k^2, k_1^2 and k_2^2 are the dimensionless parameters defined as

$$,\quad ,\quad k^2 = (d/4a)^2 \qquad k_1^2 = k^2(1+k^2) \qquad k_2^2 = k^2(1+4k^2+k^4) \tag{5}$$

for an arch with circular cross section of diameter d and

$$k^2 = (h/2a)\coth(h/2a) - 1, \qquad k_1^2 = k^2(1+k^2) + (1/3)(h/2a)^2,$$
$$k_2^2 = k^2[k^2 + k^4 + (h/2a)^2] + (1/3)(h/2a)^2 \tag{6}$$

for an arch with rectangular cross section of height h. If the arch is clamped at $\theta = 0$ and $\theta = \theta_0$, then the boundary conditions take the form

$$u(0) = \psi(0) = w(0) = u(\theta_0) = \psi(\theta_0) = w(\theta_0) = 0. \tag{7}$$

3. Differential Quadrature Method(DQM)

The differential quadrature method(DQM) was introduced by Bellman and Casti [10]. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al. [11]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Han and Kang [12] applied the method to the buckling analysis of circular curved beams. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}\, f(x_j) \quad \text{for } i,\ j=1,2,\ldots,N \tag{8}$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x) = x^{k-1} \quad \text{for} \quad k = 1,2,3,\ldots,N \tag{9}$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n)x_i^{k-n-1} \quad \text{for} \quad i,\ k = 1,2,\ldots,N \tag{10}$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming [13].

4. Application

Applying the differential quadrature method to equations (1), (2), and (3) gives

$$\frac{k}{2(1+v)}\frac{1}{a\theta_0^2}\sum_{j=1}^{N} B_{ij}\, u_j - (1-\lambda^2\frac{1+k^2}{s_y^2}\frac{1}{a}\, u_i - \frac{k}{2(1+v)}\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\, \phi_j$$

$$+[1+\frac{k}{2(1+v)}]\frac{1}{a\theta_0}\sum_{j=1}^{N} A_{ij}\, w_j = 0, \tag{11}$$

$$\frac{k}{2(1+v)}\frac{1}{a\theta_0}\sum_{j=1}^{N} A_{ij}\, u_j + k^2\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\, \phi_j - [\frac{k}{2(1+v)} - \lambda^2\frac{k_2^2}{s_y^2}]\,\phi_i$$

$$+[\frac{k}{2(1+v)} + \lambda^2\frac{k_1^2}{s_y^2}]\frac{1}{a}\, w_i = 0, \tag{12}$$

$$[1+\frac{k}{2(1+v)}]\frac{1}{a\,\theta_0}\sum_{j=1}^{N}A_{ij}\,u_j-[\frac{k}{2(1+v)}+\lambda^2\frac{k_1^2}{s_y^2}]\,\psi_i-\frac{1}{a\theta_0^2}\sum_{j=1}^{N}B_{ij}\,w_j$$

$$+[\frac{k}{2(1+v)}-\lambda^2\frac{1+k^2}{s_y^2}]\frac{1}{a}\,w_i=0 \qquad (13)$$

where A_{ij} and B_{ij} are the weighting coefficients for the first and second respectively, along the dimensionless axis.

The boundary conditions for clamped ends, given by equations (7), can be expressed in differential quadrature form as follows:

at $X=0$: $u_1=\psi_1=w_1=0$ (14)

at $X=1$: $u_N=\psi_N=w_N=0$ (15)

This set of equations together with the boundary conditions can be solved to obtain the fundamental natural frequency for in-plane vibration of a circular arch.

5. Numerical results and comparisons

The natural frequencies of vibration are calculated by the differential quadrature method The values λ corresponding to the lowest natural frequencies are evaluated for circular arches of rectangular and circular cross-sections under clamped-clamped end conditions, and numerical results are compared with transfer matrix solutions by Irie et al. [9]. The shear correction factor, k is taken to be 0.85 for the rectangular cross section and 0.89 for the circular cross-section, and the Poisson's ratio of the arch, v, is 0.3. All results are computed with thirteen discrete points along the dimensionless X-axis. The accuracy of the numerical solution increases with increasing N, passes through a maximum, but then decreases due to numerical instabilities if N becomes too large (see Han and Kang [12]). The results are summarized for in-plane vibration in Tables 1 and 2. As can be seen, the numerical results show excellent agreement with the solutions by Irie et al. [9] except 12.57 * in Table 2.

According to Irie et al. [9], the frequency parameters of rectangular-cross-section arches are generally smaller than those of circular-cross-section arches and the difference between them is very small. It seems, therefore, that 12.57 * should be 10.57.

6. Conclusions

The differential quadrature method was used to compute the eigenvalues of the equations of motion governing the free in-plane extensional and shear deformable vibrations of curved beams. The present method gives results which agree very well with the numerical solutions by other methods for the cases treated while requiring only a limited number of grid points.

Table 1 : Fundamental frequency parameter of in-plane vibration $\lambda = (\rho A\ a^4\ p^2 / E\ I_y)^{1/2}$ for clamped-clamped arches with circular cross-section ; v = 0.3

s_y	θ_0(degrees)	Irie et al [9]	DQM
20	60	23.75	23.758
	120	10.61	10.613
	180	4.151	4.1543
100	60	52.82	52.827
	120	11.79	11.793
	180	4.375	4.3757

Table 2 : Fundamental frequency parameter of in-plane vibration $\lambda = (\rho A\ a^4\ p^2 / E\ I_x)^{1/2}$ for clamped-clamped arches with circular cross-section including shear deformation; v = 0.3

s_y	θ_0(degrees)	Irie et al [9]	DQM
20	60	23.70	23.709
	120	12.57 *	10.585
	180	4.143	4.1478
100	60	52.78	52.795
	120	11.79	11.792
	180	4.374	4.3755

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