On availability of Bayesian imperfect repair model

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Abstract

Lim *et al.*(1998) proposed the Bayesian Imperfect Repair Model, in which a failed system is perfectly repaired with probability P and is minimally repaired with probability $1-P$, where P is not fixed but a random variable with a prior distribution $\Pi(p)$. In this note, the steady state availability of the model is derived and the measure is obtained for several particular prior distribution functions.

Keywords: Availability; Perfect repair; Imperfect repair; Prior distribution

1. Introduction

Consider a system which can be in one of two states, namely 'up' and 'down'. By 'up' we mean the system is still functioning and by 'down' we mean the system is not functioning; in the latter case the system is being repaired or replaced, depending on whether the component is repairable or not. Let the state of the system be given by the binary variable

$$X(t) = \begin{cases} 1 & \text{if the system is up at time } t \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

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An important characteristic of a repairable system is *availability*. The availability at time t is defined by

$$A(t) = P(X(t) = 1), \tag{2}$$

which is the probability that the system is functioning at time t. Because the study of $A(t)$ is too hard except for a few simple cases, other measures have been proposed, and more attention is being paid to the limiting behavior of this quantity. The steady state availability (or limiting availability) of the system is, *when the limit exists*, defined by

$$A = \lim_{t \to \infty} A(t), \tag{3}$$

which is a significant measure of performance of a repairable system. Some other kinds of availability which are useful in practical applications can be found in Birolini(1985, 1994) and Høyland and Rausand(1994).

In recent years various models for repairable systems with imperfect repair have been suggested. Brown and Proschan(1983) examined a maintenance action, called imperfect repair, which with probability p, is a perfect repair, and with probability $1-p$, is a minimal repair, restoring the failed system to its condition just prior to failure. Their model has been generalized by Block *et al.*(1985) to the case in which the probability of perfect repair is state-dependent, and by Shaked and Shanthikumar(1986) to the multivariate case. More recently, Lim *et al.*(1998) proposed the Bayesian Imperfect Repair Model, in which a failed system is perfectly repaired with probability P and is minimally repaired with probability $1 - P$, where P is not fixed but a random variable with a prior distribution $\Pi(p)$.

In this note, the steady state availability of the Bayesian Imperfect Repair Model is derived and the measure is obtained for several particular prior distribution functions.

2. Main result

Some notations and assumptions are described before proceeding to derive the steady state availability of the Bayesian Imperfect Repair Model.

Let $\lambda(t)$ be the failure rate function of the system and define $\Lambda(t) \equiv \int_0^t \lambda(u)du$. On each failure the system is perfectly repaired with probability P and is minimally repaired with probability $1-P$, where P is not fixed but a random variable with a prior distribution $\Pi(p)$. And it is assumed that $\Pi(0) - \Pi(0-) = 0$, i.e., $Prob(P = 0) = 0$. The probability of perfect repair P changes after each perfect repair throughout the entire process.

Define T_i as the time at which the ith perfect repair is completed, and N_i as the number of failures in the ith renewal cycle, $(T_{i-1},\ T_i]$, $i = 1, 2, \cdots$, where $T_0 \equiv 0$. Also define $X_{i,j}$ as the jth lifetime of the system in ith renewal cycle, $i = 1, 2, \cdots$, $j = 1, 2, \cdots, N_i$ and $Y_{i,j}$ as the repair time which corresponds to $X_{i,j}$. Let $F_j(x)$ and μ_j be the distribution function and the mean of $X_{i,j}$, respectively.

Assume that the minimal repair times are independent and identically distributed and the perfect repair times are also independent and identically distributed. Let $G_1(y)$ and ν_1 be the distribution function and the mean of $Y_{i,j}$, $j = 1, 2, \cdots, N_i - 1$, and $G_2(y)$ and ν_2 be the distribution function and the mean of $Y_{i,j}$, $j = N_i$. Define $Z_i \equiv T_i - T_{i-1}$, $i = 1, 2, \cdots$, then Z_i's are times between renewals and $Z_i = \sum_{j=1}^{N_i}(X_{i,j} + Y_{i,j})$. Also define $H(t)$ as the distribution function of Z_i and assume that Z_i's are mutually independent, $i = 1, 2, \cdots$.

Now the steady state availability of the system is derived. Observe that

$$\begin{aligned} A(t) &= P\{X(t) = 1\} \\ &= P\{X(t) = 1,\ t \le T_1\} + \sum_{n=1}^{\infty} P\{X(t) = 1,\ T_n < t \le T_{n+1}\}. \end{aligned}$$

Let P_i be the perfect repair probability in the ith renewal cycle. Also define $Z_{i,j} \equiv \sum_{m=1}^{j}(X_{i,m} + Y_{i,m})$ and $F_{Z_{i,j}}(t)$ as its distribution function, $i = 1, 2, \cdots$, $j = 1, 2, \cdots, N_i - 1$, then

$$A_0(t) \equiv P\{X(t) = 1,\ t \le T_1\}$$

$$
\begin{aligned}
&= \int P\{X(t)=1,\ t\le T_1 |\ P_1=p\}d\Pi(p)\\
&= \int \Big\{ \sum_{r=1}^{\infty} P\{X(t)=1,\ t\le T_1 |\ P_1=p, N_1=r\}\\
&\qquad\qquad \times P\{N_1=r|\ P_1=p\}\Big\} d\Pi(p)\\
&= \int \Big\{ p\bar{F}_1(t) + \sum_{r=2}^{\infty} \Big[\bar{F}_1(t) + \sum_{j=1}^{r-1} P\{Z_{1,j} < t \le Z_{1,j} + X_{1,j+1}\}\Big]\\
&\qquad\qquad \times p(1-p)^{r-1}\Big\} d\Pi(p)\\
&= \int p\bar{F}_1(t)d\Pi(p) + \int \sum_{r=2}^{\infty} p(1-p)^{r-1}\Big\{\bar{F}_1(t)\\
&\qquad\qquad + \sum_{j=1}^{r-1}\int_0^t \bar{F}_{j+1|Z_{1,j}=s}(t-s)dF_{Z_{1,j}}(s)\Big\} d\Pi(p),
\end{aligned}
$$

where $\bar{F}_{j+1|Z_{i,j}=s}(t)$ is the conditional survivor function of $X_{i,j+1}$ given $Z_{i,j}=s$, that is, $P\{X_{i,j+1} \ge t \mid Z_{i,j}=s\}$, $i=1,2,\cdots,\ j=1,2,\cdots,N_i-1$.

Furthermore if we define $H^{(n)}(t)$ as the n-fold convolution of $H(t)$ and $M_H(t)$ as $\sum_{n=1}^{\infty} H^{(n)}(t)$ then,

$$
\begin{aligned}
&P\{X(t)=1,\ T_n < t \le T_{n+1}\}\\
&= \int \Big\{ \sum_{r=1}^{\infty} P\{X(t)=1,\ T_n < t \le T_{n+1}|\ P_{n+1}=p,\ N_{n+1}=r\}\\
&\qquad\qquad \times P\{N_{n+1}=r|\ P_{n+1}=p\}\Big\} d\Pi(p)\\
&= \int p\cdot P\{T_n < t \le T_n + X_{n+1,1}\}d\Pi(p)\\
&\quad + \int \Big\{ \sum_{r=2}^{\infty} \Big[\ P\{T_n < t \le T_n + X_{n+1,1}\}\\
&\qquad + \sum_{j=1}^{r-1} P\{T_n + Z_{n+1,j} < t \le T_n + Z_{n+1,j} + X_{n+1,j+1}\}\ \Big]\\
&\qquad\qquad \times p(1-p)^{r-1}\Big\} d\Pi(p)\\
&= \int\int_0^t p\bar{F}_1(t-u)dH^{(n)}(u)d\Pi(p)
\end{aligned}
$$

$$+\int \sum_{r=2}^{\infty} p(1-p)^{r-1}\Bigg[\int_0^t \Bigg\{\bar{F}_1(t-u)$$

$$+\sum_{j=1}^{r-1}\int_0^{t-u} \bar{F}_{j+1|Z_{n+1,j}=s}(t-u-s)dF_{Z_{n+1,j}}(s)\Bigg\}dH^{(n)}(u)\Bigg]d\Pi(p)$$

$$= \int_0^t \int p\bar{F}_1(t-u)d\Pi(p)dH^{(n)}(u)$$

$$+\int_0^t \int \Bigg[\sum_{r=2}^{\infty} p(1-p)^{r-1}\Bigg\{\bar{F}_1(t-u)$$

$$+\sum_{j=1}^{r-1}\int_0^{t-u} \bar{F}_{j+1|Z_{n+1,j}=s}(t-u-s)dF_{Z_{n+1,j}}(s)\Bigg\}\Bigg]d\Pi(p)dH^{(n)}(u)$$

$$= \int_0^t A_0(t-u)dH^{(n)}(u), \text{ by the monotone convergence theorem.}$$

Hence,

$$\begin{aligned} A(t) &= A_0(t) + \sum_{n=1}^{\infty}\int_0^t A_0(t-u)dH^{(n)}(u) \\ &= A_0(t) + \int_0^t A_0(t-u)dM_H(u). \end{aligned}$$

Note that $\lim_{t\to\infty} A_0(t) \le \lim_{t\to\infty} P\{T_1 \ge t\} = 0$ and, if A_0 is directly Riemann integrable, by the Key Renewal Theorem,

$$\begin{aligned} \lim_{t\to\infty} A(t) &= \lim_{t\to\infty}\int_0^t A_0(t-u)dM_H(u) \\ &= \frac{1}{E(Z_i)}\int_0^{\infty} A_0(t)dt, \end{aligned}$$

where if we define $V_i \equiv \sum_{j=1}^{N_i} X_{i,j}$ and $W_i \equiv \sum_{j=1}^{N_i} Y_{i,j}$,

$$\begin{aligned} E(Z_i) &= \int E(V_i + W_i|\ P_i = p)d\Pi(p) \\ &= \int E(V_i|\ P_i = p)d\Pi(p) + \int \frac{1-p}{p}\nu_1 d\Pi(p) + \nu_2 \\ &= \int\int_0^{\infty} \exp(-p\Lambda(v))dv d\Pi(p) + \int \frac{1-p}{p}\nu_1 d\Pi(p) + \nu_2. \end{aligned}$$

On the other hand, by the fact that

$$\int_0^{\infty}\int_0^t \bar{F}_{j+1|Z_{1,j}=s}(t-s)dF_{Z_{1,j}}(s)dt = \mu_{j+1},$$

we obtain

$$\int_0^\infty A_0(t)dt = \int\int_0^\infty \exp(-p\Lambda(v))dvd\Pi(p).$$

Therefore the steady state availability of the Bayesian Imperfect Repair Model exists and is given by

$$A = \frac{\int\int_0^\infty \exp(-p\Lambda(v))dvd\Pi(p)}{\int\int_0^\infty \exp(-p\Lambda(v))dvd\Pi(p) + \int \frac{1-p}{p}\cdot\nu_1 d\Pi(p) + \nu_2}. \tag{4}$$

Note that the steady state availability given in (4) depends only on the mean of total up time in a renewal cycle and that of total down time in a renewal cycle, i.e., $E(V_i)$ and $E(W_i)$.

3. Particular cases

In this section the special case when the system failure rate is a Weibull failure rate, which is defined by

$$\lambda(t) = \lambda\beta t^{\beta-1},\ \lambda > 0,\ \beta > 0,$$

is considered. In this case,

$$\begin{aligned} E(V_i|\ P_i = p) &= \int_0^\infty \exp(-p\Lambda(v))dv \\ &= \frac{1}{(p\lambda)^{\frac{1}{\beta}}}\cdot\Gamma(\frac{1}{\beta}+1). \end{aligned}$$

As particular prior distributions the Uniform distribution and the Beta distribution are assumed. Let the corresponding pdf of $\Pi(p)$ be $\pi(p)$.

(I) $Uniform(\alpha_1, \alpha_2)$ Prior;

$$\pi(p) = \frac{1}{(\alpha_2-\alpha_1)}I_{(\alpha_1,\alpha_2)}(p),\ 0 \le \alpha_1 < \alpha_2 \le 1.$$

When $\beta > 1$ and $0 \le \alpha_1 < \alpha_2 \le 1$ since

$$\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{1}{(p\lambda)^{\frac{1}{\beta}}}\cdot\Gamma(\frac{1}{\beta}+1)dp =$$

$$\frac{\beta}{(\beta-1)(\alpha_2-\alpha_1)}\lambda^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)(\alpha_2^{\frac{\beta-1}{\beta}} - \alpha_1^{\frac{\beta-1}{\beta}})$$

and

$$\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{1-p}{p}dp=\{\frac{1}{\alpha_2-\alpha_1}(\ln(\alpha_2)-\ln(\alpha_1))-1\},$$

the steady state availability is given by

$$\begin{aligned} A \ = \ & \left[\frac{\beta}{(\beta-1)(\alpha_2-\alpha_1)}\lambda^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)(\alpha_2^{\frac{\beta-1}{\beta}}-\alpha_1^{\frac{\beta-1}{\beta}})\right]\Big/\left[\frac{\beta}{(\beta-1)(\alpha_2-\alpha_1)}\right. \\ & \left.\times\lambda^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)(\alpha_2^{\frac{\beta-1}{\beta}}-\alpha_1^{\frac{\beta-1}{\beta}})+\left(\frac{1}{\alpha_2-\alpha_1}\ln(\frac{\alpha_2}{\alpha_1})-1\right)\cdot\nu_1+\nu_2\right]. \quad (5)\end{aligned}$$

Note that if $\alpha_1=0$ in this case, $A=0$. This somewhat interesting result arises from the following facts. The random variable N_i assumes large values with probabilities large enough that $E(N_i)=\infty$. Thus the mean of total down time in a renewal cycle is infinite. But on the other hand, since the system is deteriorating($\beta>1$), the mean of total up time in a renewal cycle is finite.

When $\beta=1$ and $0<\alpha_1<\alpha_2\le 1$,

$$A=\frac{\frac{\lambda^{-1}}{\alpha_2-\alpha_1}\ln(\frac{\alpha_2}{\alpha_1})}{\frac{\lambda^{-1}}{\alpha_2-\alpha_1}\ln(\frac{\alpha_2}{\alpha_1})+\left(\frac{1}{\alpha_2-\alpha_1}\ln(\frac{\alpha_2}{\alpha_1})-1\right)\cdot\nu_1+\nu_2}. \quad (6)$$

When $\beta<1$ and $0<\alpha_1<\alpha_2\le 1$, the steady state availability A is also given by (5).

(II) $Beta(\alpha_1,\alpha_2)$ Prior;

$$\pi(p)=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}p^{(\alpha_1-1)}(1-p)^{(\alpha_2-1)},\ \alpha_1,\alpha_2>0,\ 0\le p\le 1.$$

Observe that if $\alpha_1>1/\beta$,

$$\begin{aligned}&\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1\frac{1}{(p\lambda)^{\frac{1}{\beta}}}\cdot\Gamma(\frac{1}{\beta}+1)p^{(\alpha_1-1)}(1-p)^{(\alpha_2-1)}dp\\ &\qquad=\lambda^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)\frac{\Gamma(\alpha_1-\frac{1}{\beta})\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_1-\frac{1}{\beta}+\alpha_2)},\end{aligned}$$

and if $\alpha_1>1$,

$$\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1\frac{1-p}{p}p^{(\alpha_1-1)}(1-p)^{(\alpha_2-1)}dp=\frac{\alpha_2}{\alpha_1-1}.$$

Therefore the steady state availability is given by

$$A = \frac{\lambda^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)\frac{\Gamma(\alpha_1-\frac{1}{\beta})\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_1-\frac{1}{\beta}+\alpha_2)}}{\lambda^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)\frac{\Gamma(\alpha_1-\frac{1}{\beta})\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_1-\frac{1}{\beta}+\alpha_2)}+\frac{\alpha_2}{\alpha_1-1}\cdot\nu_1+\nu_2}, \tag{7}$$

if $\alpha_1 > \max(1/\beta,\ 1)$.

Also note that when $1/\beta < \alpha_1 \leq 1$ the steady state availability $A = 0$. In this case, the prior distribution also enables P_i to take suitably small values so that the mean of total down time is infinite, whereas that of total up time in a renewal cycle is finite.

Values for steady state availability are given in Table 1 for different Beta prior distributions when $\lambda = 1.0$ and $\nu_2 = 0.1$. The parameter β values, which represent the degree of system improvement or deterioration, used are: $\beta = 2/3,\ 4/5,\ 1.0,\ 3.0,\ 5.0$ and the parameter ν_1 values used are: $\nu_1 = 0.01,\ 0.05,\ 0.45$. The Beta prior distributions used are: $Beta(4,1)$, $Beta(3,1)$, $Beta(2,1)$, $Beta(2,2)$, $Beta(2,3)$ and $Beta(2,4)$. If the corresponding distribution functions are denoted by $\Pi_1(p)$, $\Pi_2(p)$, $\Pi_3(p)$, $\Pi_4(p)$, $\Pi_5(p)$ and $\Pi_6(p)$, respectively, then the random variables U_i from the prior distributions Π_i, $U_i \sim \Pi_i(p)$, $i = 1, 2, \cdots, 6$, have the following usual stochastic orderings;

$$U_{i+1} \leq_{st} U_i, \text{ for } i = 1, 2, \cdots, 5, \tag{8}$$

which mean that as the prior distribution changes from Π_1 to Π_6, the repair by the prior distribution is more and more likely to be a minimal one.

As the prior distribution changes from Π_1 to Π_6, the mean of total up time and that of total down time in a renewal cycle increase together. Thus the trend changes of availability values given in Table 1 depend on the relative sizes of changing increments related with the mean of total up time and that of total down time in a renewal cycle. In particular, when the system is improving($\beta < 1$), as the prior changes from Π_1 to Π_6, the mean of total up time increases more and more rapidly. Conversely, for the cases of deteriorating system($\beta > 1$), it

Table 1: Availability values for several Beta prior distributions

ν_1	β	$\Pi_1(p)$	$\Pi_2(p)$	$\Pi_3(p)$	$\Pi_4(p)$	$\Pi_5(p)$	$\Pi_6(p)$
	2/3	0.953668	0.962007	0.979732	0.988842	0.992418	0.994274
	4/5	0.940997	0.948713	0.964871	0.977356	0.983037	0.986254
0.01	1.0	0.928074	0.934579	0.947867	0.961538	0.968523	0.972763
	3.0	0.904098	0.905371	0.906904	0.909470	0.910042	0.909622
	5.0	0.903411	0.903559	0.902671	0.901076	0.898484	0.895407
	2/3	0.948001	0.955095	0.972564	0.981541	0.985520	0.987809
	4/5	0.933888	0.939534	0.952701	0.962821	0.967882	0.971000
0.05	1.0	0.919540	0.923077	0.930233	0.937500	0.941176	0.943396
	3.0	0.893047	0.889342	0.877208	0.857704	0.840267	0.824464
	5.0	0.892290	0.887261	0.871815	0.845328	0.821503	0.799804
	2/3	0.894823	0.891074	0.906261	0.914051	0.921475	0.927503
	4/5	0.868283	0.856657	0.845996	0.838173	0.838597	0.840937
0.45	1.0	0.842105	0.821918	0.784314	0.750000	0.733945	0.724638
	3.0	0.795778	0.755566	0.660824	0.546593	0.475610	0.425818
	5.0	0.794491	0.751672	0.649723	0.522230	0.442434	0.386806

increases more and more slowly. These explain the trend changes of availability values for the ordered Beta prior distributions presented in Table 1.

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