# 퍼지 준-연속사상에 관하여

# ON FUZZY QUASI-CONTINUOUS MAPPINGS

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ABSTRACT

The aim of this paper is to continue the study of fuzzy quasi-continuous mappings due to Park et al. [12] on fuzzy bitopological spaces.

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## 1. Introduction and preliminaries

Chang [2] used the concept of fuzzy sets to introduce fuzzy topological spaces and several authors continued the investigation of such spaces. From the fact that there are some non-symmetric fuzzy topological structures, Kubiak [8] first introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological space, and initiated the bitopological aspects due to Kelly [7] in the theory of fuzzy topological spaces. Since then several authors [3,4,6,8-10,12] have contributed to the subsequent development of various fuzzy bitopological properties. Recently, Park et al. [12] defined and studied fuzzy quasi-open sets and fuzzy quasi-continuous mappings on fuzzy bitopological spaces. The aim of this paper is to continue the study of fuzzy quasi-continuous mappings between fuzzy bitopological spaces.

For definitions and results not explained in this paper, we refer to the papers [2,11-13] assuming them to be well known. A fuzzy point in $X$ with support $x \in X$ and value $\alpha\, (0 < \alpha \leq 1)$ is denoted by $x_\alpha$. For a fuzzy set $A$ of $X$, $1-A$ will stand for the complement of $A$. By $0_X$ and $1_X$ we will mean respectively the constant fuzzy sets taking on the values 0 and 1 on $X$.

A system $(X, \tau_1, \tau_2)$ consisting of a set $X$ with two topologies $\tau_1$ and $\tau_2$ on $X$ is called a fuzzy bitopological space [7] (for short, fbts). A fuzzy set $A$ of a fbts $(X, \tau_1, \tau_2)$ is called fuzzy quasi-open [12] (briefly, fqo) if for each fuzzy point $x_\alpha \in A$ there is either a $U \in \tau_1$ such that $x_\alpha \in U \leq A$, or a $V \in \tau_2$ such that $x_\alpha \in V \leq A$. A fuzzy set $A$ is fuzzy quasi-closed (briefly, fqc) if the complement $1-A$ is a fqo set. A fuzzy set $A$ of a fbts $(X, \tau_1, \tau_2)$ is called a quasi-Q-nbd [12] (resp. quasi-nbd [12]) of a fuzzy point $x_\alpha$ if there exists a fqo set $U$ such that $x_\alpha \mathrm{q} U \leq A$ (resp. $x_\alpha \in U \leq A$).

**Result 1 [12].** A fuzzy set $A$ of a fbts $(X, \tau_1, \tau_2)$ is fqo if and only if there exist $U \in \tau_1$ and $V \in \tau_2$ such that $A = U \cup V$.

**Result 2 [12].** Let $A$ be any fuzzy set of a fbts $X$. Then $x_\alpha \in \mathrm{qcl}(A)$ if and only if for each fqo quasi-Q-nbd $U$ of $x_\alpha$, $U \mathrm{q} A$.

## 2. Some properties of fuzzy quasi-continuous mappings

A mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be fuzzy quasi-continuous [12] if $f^{-1}(B)$ is fqo in $X$ for each $B\in\sigma_i$, equivalently, $f^{-1}(B)$ is fqc in $X$ for each $\sigma_i$-fc set $B$ of $Y$.

**Theorem 2.1.** For a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following are equivalent:
(a) $f$ is fuzzy quasi-continuous.
(b) $f^{-1}(V)$ is fqo in $X$ for each fqo set $V$ of $Y$.

**Proof.** (a)$\Rightarrow$(b): Let $V$ be any fqo set of $Y$. By result 1, there exist $V_1\in\sigma_1$ and $V_2\in\sigma_2$ such that $V=V_1\cup V_2$. Since $f$ is fuzzy quasi-continuous, $f^{-1}(V)=f^{-1}(V_1\cup V_2)$ $=f^{-1}(V_1)\cup f^{-1}(V_2)$ is fqo in $X$.
(b)$\Rightarrow$(a): Let $V\in\sigma_i$. Since every $\sigma_i$-fuzzy open set is fqo, $f^{-1}(V)$ is fqo in $X$. Hence $f$ is fuzzy quasi-continuous.

**Theorem 2.2.** For a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following are equivalent:
(a) $f$ is fuzzy quasi-continuous.
(b) For each fuzzy point $x_\alpha$ in $X$ and each fqo quasi-nbd $V$ of $f(x_\alpha)$, there exists a fqo quasi-nbd $U$ of $x_\alpha$ such that $f(U)\le V$.
(c) For each fuzzy point $x_\alpha$ in $X$ and each quasi-Q-nbd $V$ of $f(x_\alpha)$, there 1s a quasi-Q-nbd $U$ of $x_\alpha$ such that $f(U)\le V$.
(d) $f(\mathrm{qcl}(A))\le \mathrm{qcl}(f(A))$ for each fuzzy set $A$ of $X$.
(e) $\mathrm{qcl}(f^{-1}(B))\le f^{-1}(\mathrm{qcl}(B))$ for each fuzzy set $B$ of $Y$.

**Proof.** (a)$\Rightarrow$(b): Let $x_\alpha$ be any fuzzy point in $X$ and $V$ be any fqo quasi-nbd of $f(x_\alpha)$. Then $f^{-1}(V)=U$ (say) is a fqo quasi-nbd of $x_\alpha$ such that $f(U)\le V$.
(b)$\Rightarrow$(c): Let $x_\alpha$ be any fuzzy point in $X$ and $V$ be any fqo quasi-Q-nbd of $f(x_\alpha)$. Since $V(f(x))+\alpha>1$, there exists a real number $\beta>0$ such that $V(f(x))>\beta>1-\alpha$, so that $V$ is fqo quasi-nbd of $f(x)_\beta$. By (b), there is a fqo quasi-nbd $U$ of $x_\beta$ such that $f(U)\le V$. Now, $U(x)\ge\beta$ implies $U(x)>1-\alpha$ and thus $U$ is a fqo quasi-Q-nbd of $x_\alpha$.
(c)$\Rightarrow$(d): Suppose that $x_\alpha\in \mathrm{qcl}(A)$ such that $f(x_\alpha)\notin \mathrm{qcl}(f(A))$. Then there is a fqo quasi-Q-nbd $V$ of $f(x_\alpha)$ such that $V \not q f(A)$ which implies $A\le 1-f^{-1}(V)$. By (c), there exists a fqo quasi-Q-nbd of $U$ of $x_\alpha$ such that $U\le f^{-1}(V)$. Now, we have
$A\le 1-f^{-1}(V)\Rightarrow A\le 1-U\Rightarrow A\not q U$.
This a contradiction since $x_\alpha\in \mathrm{qcl}(A)$.
(d)$\Rightarrow$(e): Let $B$ be a fuzzy set of $Y$. Then by (d) we have
$f(\mathrm{qcl}(f^{-1}(B)))\le \mathrm{qcl}(f(f^{-1}(B)))\le \mathrm{qcl}(B)$
and thus $\mathrm{qcl}(f^{-1}(B))\le f^{-1}(\mathrm{qcl}(B))$.
(e)$\Rightarrow$(a): Let $B$ be any fqc set of $Y$. By (e), $\mathrm{qcl}(f^{-1}(B))\le f^{-1}(\mathrm{qcl}(B))=f^{-1}(B)$ and so $f^{-1}(B)$ is fqc in $X$. Hence by Theorem 3.1 $f$ is fuzzy quasi-continuous.

**Theorem 2.3.** Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be one to one and onto. Then $f$ is fuzzy quasi-continuous if and only if $f(\mathrm{qint}(A))\le \mathrm{qint}(f(A))$ for each fuzzy set $A$ of $X$.

**Proof.** Let $A$ be any fuzzy set of $X$. Then clearly $f^{-1}(\mathrm{qint}(f(A)))$ is fqo in $X$. Since $f$ is one to one, we have
$f^{-1}(\mathrm{qint}(f(A)))\le \mathrm{qint}(f^{-1}(f(A)))=\mathrm{qint}(A)$
and $f(f^{-1}(\mathrm{qint}(f(A))))\le f(\mathrm{qint}(A))$.
Since $f$ is onto, we have
$\mathrm{qint}(f(A))=f(f^{-1}(\mathrm{qint}(f(A))))\le f(\mathrm{qint}(A))$
Conversely, let $B$ be a fqo set of $Y$. Then $f(\mathrm{qint}(f^{-1}(B)))\ge \mathrm{qint}(f(f^{-1}(B)))=B$ and thus $f^{-1}(f(\mathrm{qint}(f^{-1}(B))))\ge f^{-1}(B)$. Since $f$ is one to one, $\mathrm{qint}(f^{-1}(B))\ge f^{-1}(B)$. This shows that $f^{-1}(B)$ is fqo in $X$. Hence $f$ is fuzzy quasi-continuous.

**Theorem 2.4.** Let $(X,\tau_1,\tau_2)$ and $(Y,\delta_1,\delta_2)$ be fbts's. If the graph mapping $g:(X,\tau_1,\tau_2)\to(X\times Y,\delta_1,\delta_2)$ of $f$, where $\delta_i$ is the fuzzy product topology generated by $\tau_i$ and $\sigma_i$ (for $i=1,2$), defined by $g(x)=(x,f(x))$ for each fuzzy quasi-continuous.

**Proof.** Let $V$ be any fqo set of $Y$. Then by Lemma 2.4 in [1], we have

$$f^{-1}(V)=1\cap f^{-1}(V)=g^{-1}(1\times V).$$

Since $1\times V$ is $\delta_i$-fo set of $X\times Y$ and $g$ is fuzzy quasi-continuous, $f^{-1}(V)$ is fqo set of $X$.

**Remark 2.5.** For a fbts $(X,\tau_1,\tau_2)$, we have $FQT_2\Rightarrow FQT_1\Rightarrow FQT_0$ [12].

**Theorem 2.6.** Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be one to one. If $f$ is fuzzy quasi-continuous and $(Y,\sigma_1,\sigma_2)$ is $FQT_k$, then $(X,\tau_1,\tau_2)$ is $FQT_k$, for $k=0,1,2$ .

**Proof.** We give a proof for $k=1$ only; the other cases, being similar, are left. Let $x_\alpha$ and $y_\beta$ be two distinct fuzzy points in $X$.
When $x\neq y$, $f(x)\neq f(y)$. Since $(Y,\sigma_1,\sigma_2)$ is $FQT_1$, there exist quasi-nbds $U$ and $V$ of $f(x_\alpha)$ and $f(y_\beta)$ respectively such that $f(x_\alpha)\not q V$ and $f(y_\beta)\not q U$. Since $f$ is fuzzy quasi-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are quasi-nbds of $x_\alpha$ and $y_\beta$ respectively such that $x_\alpha \not q f^{-1}(V)$ and $y_\beta \not q f^{-1}(U)$.
When $x=y$ and $\alpha<\beta$ (say), then $f(x)=f(y)$. Since $(Y,\sigma_1,\sigma_2)$ is $FQT_1$, there exists a quasi-Q-nbd $V$ of $f(y_\beta)$ such that $f(x_\alpha)\not q V$. Then $f^{-1}(V)$ is quasi-Q-nbd of $y_\beta$ such that $x_\alpha \not q f^{-1}(V)$. Hence $(X,\tau_1,\tau_2)$ is $FQT_1$.

A fuzzy set $A$ of a fbts $(X,\tau_1,\tau_2)$ which can not be expressed as the union of two fuzzy quasi-separated sets is said to be a fuzzy quasi-connected set [12].

**Theorem 2.7.** Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a fuzzy quasi-continuous onto mapping. If $A$ is fuzzy quasi-connected in $(X,\tau_1,\tau_2)$, then $f(A)$ is fuzzy quasi-connected in $(Y,\sigma_1,\sigma_2)$.

**Proof.** Suppose that $f(A)$ is not fuzzy quasi-connected in $(Y,\sigma_1,\sigma_2)$. Then there exist fuzzy quasi-separated sets $B$ and $C$ in $Y$ such that $f(A)=B\cup C$. There exist fqo subsets $U$ and $V$ such that $B\leq U$, $C\leq V$, $B\not q V$ and $C\not q U$. Since $f$ is fuzzy quasi-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are fqo in $X$ and $A=f^{-1}(f(A))=f^{-1}(B\cup C)=f^{-1}(B)\cup f^{-1}(C)$. Also it can be easily seen that $f^{-1}(B)$ and $f^{-1}(C)$ are fuzzy quasi-separated in $X$. Thus we arrive at a contradiction.

A mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be fuzzy quasi-open [12] (briefly, fq open) (resp. fuzzy quasi-closed [12] (briefly, fq closed)) if $f(U)$ is fqo (resp. fqc) in $Y$ for each $\tau_i$-fuzzy open (resp. $\tau_i$-fuzzy closed) set $U$ of $X$.

**Theorem 2.8.** For a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following are equivalent:
(a) $f$ is fq open.
(b) $f(A)$ is fqo in $Y$ for each fqo set $A$ of $X$.
(c) $f(\operatorname{qint}(A))\leq \operatorname{qint}(f(A))$ for each fuzzy set $A$ of $X$.
(d) $\operatorname{qint}(f^{-1}(B))\leq f^{-1}(\operatorname{qint}(B))$ for any fuzzy set $B$ of $Y$.

**Proof.** (a)$\Rightarrow$(b): Let $A$ be any fqo set of $X$. Then there exist $U\in\tau_1$ and $V\in\tau_2$ such that $A=U\cup V$. Since $f$ is fq open, $f(A)=f(U\cup V)=f(U)\cup f(V)$ is fqo in $Y$.
(b)$\Rightarrow$(a): Straightforward.
(b)$\Rightarrow$(c): Let $A$ be any fuzzy set of $X$. Then by (b) $f(\operatorname{qint}(A))$ is fqo in $Y$ and hence $f(\operatorname{qint}(A))\leq \operatorname{qint}(f(A))$.
(c)$\Rightarrow$(b): Let $A$ be any fqo set of $X$. Then $f(A)=f(\operatorname{qint}(A))\leq \operatorname{qint}(f(A))\leq f(A)$ and so $f(A)$ is fqo in $Y$.
(c)$\Leftrightarrow$(d): Straightforward.

**Theorem 2.9.** If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is fq open, then for each fuzzy set $B$ of $Y$ and each fqc set $A$ of $X$ such that $f^{-1}(B)\leq A$, there is a fqc set $C$ of $Y$ such that $B\leq C$ and $f^{-1}(C)\leq A$.

**Proof** Let $B$ be a fuzzy set of $Y$ and $A$ be a fqc set of $X$ such that $f^{-1}(B)\leq A$. Since a fqc set of $X$ such that $f^{-1}(B) \leq A$. Since $f$ is fq open and $1-A$ is fqo in $X$, $f(1-A)$ is fqo in $Y$ and thus $f(1-A) \leq \mathrm{qint}(1-B) = 1 - \mathrm{qcl}(B)$, i.e. $f^{-1}(\mathrm{qcl}(B)) \leq A$. Put $C = \mathrm{qcl}(B)$. Then $C$ is a fqc set of $Y$ such that $B \leq C$ and $f^{-1}(C) \leq A$.

**Theorem 2.10.** Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be fbts's and $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be one to one and onto. Then $f$ is fq closed if and only if $f^{-1}(\mathrm{qcl}(B)) \leq \mathrm{qcl}(f^{-1}(B))$ for each fuzzy set $B$ in $Y$.

**Proof.** Straightforward.

**Theorem 2.11.** Let $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g:(Y, \sigma_1, \sigma_2) \to (Z, \delta_1, \delta_2)$ be mappings.

(a) If $f$ and $g$ are fuzzy quasi-continuous, then $g \circ f$ is fuzzy quasi-continuous.

(b) If $g \circ f$ is fq open and $f$ is fuzzy quasi-continuous and onto, then $g$ is fq open.

(c) If $g \circ f$ is fq open and $g$ is fuzzy quasi-continuous and one to one, then $f$ is fq open.

**Proof** Straightforward.

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