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Fuzzy Continuous Mappings and Fuzzy Set-Valued Mappings

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Abstract

First, we study some properties of F-continuities. Second, we introduce the concept of fuzzy set-valued mappings and study some properties of fuzzy set-valued mappings and fuzzy set-valued continuous mappings. Finally, we introduce the concept of fuzzy semi-continuous of fuzzy set-valued mappings and investigate their some properties.

Key words and phrases : fuzzy set-valued mapping, F-continuities of fuzzy set-valued mapping.

1. Preliminaries.

We list some concepts and results needed in the later sections.

Definition 1.1[1]. Let be a mapping, let $A\in I^X$ and let $B\in I^Y$. Then :

(1) The *inverse image* of B *under* f, denoted by $f^{-1}(B)$ is a fuzzy set in X defined by for each $x\in X$,

$$[f^{-1}(B)](x)=B(f(x))=(B\circ f)(x)$$

(2) The *image* of A *under* f, denoted by $f(A)$ is a fuzzy set in Y defined by for each $y\in Y$,

$$f(A)=\begin{cases} sup_{y=f(x)}A(x), & \text{if } y\in f(X), \\ 0, & \text{if } y\notin f(X). \end{cases}$$

By the above definition, $f: I^X\to I^Y$ and $f^{-1}: I^Y\to I^X$ are mappings.

Result 1.A[1,Theorem 4.1:9,Theorem 4.1]. Let $f: X\to Y$ be a mapping, let $\{A_\alpha\}_{\alpha\in\Lambda}\subset I^X$ and let $\{B_\alpha\}_{\alpha\in\Lambda}\subset I^Y$. Then :

(1) $f^{-1}(\bigcup_{\alpha\in\Lambda}B_\alpha)=\bigcup_{\alpha\in\Lambda}f^{-1}(B_\alpha)$

$f^{-1}(\bigcap_{\alpha\in\Lambda}B_\alpha)=\bigcap_{\alpha\in\Lambda}f^{-1}(B_\alpha)$

(2) $f(\bigcup_{\alpha\in\Lambda}A_\alpha)=\bigcup_{\alpha\in\Lambda}f(A_\alpha)$

$f(\bigcap_{\alpha\in\Lambda}A_\alpha)\subset\bigcap_{\alpha\in\Lambda}f(A_\alpha)$

(3) $f(f^{-1}(B))\subset B$ for each $B\in I^Y$.
In particular, if f is surjective, then $f(f^{-1}(B))=B$

(4) $A\subset f^{-1}(f(A))$ for each $A\in I^X$.
In particular, if f is injective, then $f^{-1}(f(A))=A$

(5) Let $g: Y\to Z$ be a mapping. If $B\in I^Z$, then $(g\circ f)^{-1}(B)=f^{-1}(g^{-1}(B))$
If $A\in I^X$, then $(g\circ f)(A)=g(f(A))$

(6) If f is bijective, then $[f(A)]^c=f(A^c)$ for each $A\in I^X$.

Proposition 1.2. Let $f: X\to Y$ be a mapping, let $A\in I^X$ and let $B\in I^Y$. Then :

(1) $f(A)=\emptyset$ if and only if $A=\emptyset$

(2) $f(A)\cap B=f(A\cap f^{-1}(B))$

Result 1.B[9,Proposition 4.2]. Let $f: X\to Y$ mapping and let $x_\lambda\in F_p(X)$.

(1) If for each $B\in I^Y$, $f(x_\lambda)qB$, then $x_\lambda qf^{-1}(B)$.

(2) If for each $A\in I^X$, $x_\lambda qf(A)$.

Result 1.C[7,Lemma 4.1]. For sets X

and Y, let $f: X \to Y$ a mapping and let $A, B \in I^X$. If AqB, then $f(A)qf(B)$.

Definition 1.3[1]. A mapping $f: (X, T) \to (Y, \mathcal{U})$ is said to be *fuzzy continuous* (in short, *F-continuous*) if $f^{-1}(B) \in T$ for each $B \in \mathcal{U}$. The mapping f is called a *fuzzy homeomorph -ism* (in short, *F-homeomorphism*) if f is bijective, and both f and f^{-1} are F-continuous.

Result 1.D[6,Result 3.1]. Let X and Y be fts's. Then a mapping $f: X \to Y$ is F-continuous if and only if for each $x_\lambda \in F_p(X)$ and each $V \in \mathcal{N}_F(f(x_\lambda))$, there is $U \in \mathcal{N}_F(f(x_\lambda))$ such that $f(U) \subset V$.

Result 1.E[2,Theorem 2.7]. Let X and Y be fts's. Then a mapping $f: X \to Y$ is F-continuous if and only if for each $x_\lambda \in F_p(X)$ and each open q-nbd V of $f(x_\lambda)$ in Y, there exists an open q-nbd U of x_λ in X such that $f(U) \subset V$.

Result 1.F[5,Theorem 1.1]. Let (X, T_X) and (Y, T_Y) be fts's and let $f: X \to Y$ a mapping. Then the following are equivalent:

(1) f is F-continuous.

(2) For each $B \in FC(Y)$, $f^{-1}(B) \in FC(X)$

(3) For each $V \in \mathcal{S}$, $f^{-1}(V) \in T_X$ where $\mathcal{S}$ is a subbase for T_Y.

(4) For each $x_\lambda \in F_p(X)$ and each $V \in \mathcal{N}_F(f(x_\lambda))$, there is $U \in \mathcal{N}_F(x_\lambda)$ such that $f(U) \subset V$.

(5) For each $x_\lambda \in F_p(X)$ and each $V \in \mathcal{N}_Q(f(x_\lambda))$, there is $U \in \mathcal{N}_Q(x_\lambda)$ such that $f(U) \subset V$.

(6) For each fuzzy net $s = \{s_n\}_{n \in D}$ if s converges to x_λ, then $f \circ s = \{f(s_n)\}_{n \in D}$ is a fuzzy net in Y and converges to $f(x_\lambda)$.

(7) For each $A \in I^X$, $f(clA) \subset clf(A)$

(8) For each $B \in I^Y$, $clf^{-1}(B) \subset f^{-1}(clB)$

From Result 1.F, we obtain the following result :

Proposition 1.4. Let X and Y be fts's, let $f: X \to Y$ a mapping and let $x_\lambda \in F_p(X)$.
Then the following are equivalent :

(1) f is F-continuous at x_λ.

(2) $x_\lambda \in f^{-1}(intB) \Rightarrow x_\lambda \in intf^{-1}(B)$ for each $B \in I^Y$.

(3) $x_\lambda \in clf^{-1}(B) \Rightarrow x_\lambda \in f^{-1}(clB)$ for each $B \in I^Y$.

Definition 1.5[8]. Let X and Y be fts's. Then a mapping $f: X \to Y$ said to be :

(1) *fuzzy open* (in short, *F-open*) if for each $U \in FO(X)$, $f(U) \in FO(Y)$.

(2) *fuzzy closed* (in short, *F-closed*) if for each $U \in FC(X)$, $f(U) \in FC(Y)$

Result 1.G[9,Theorem 4.3]. If $f: X \to Y$ is F-open, then $f^{-1}(clB) \subset clf^{-1}(B)$ for each $B \in I^Y$.

Corollary 1.G[9,Corollary 4.4]. If $f: X \to Y$ is F-open and F-continuous, then $f^{-1}(clB) = clf^{-1}(B)$ for each $B \in I^Y$.

Result 1.H[1]. If $f: X \to Y$, $g: Y \to Z$ are F- continuous, then $g \circ f: X \to Z$ is F-continuous.

2. Fuzzy-continuous mappings.

Proposition 2.1. A mapping $f: X \to Y$ is F-continuous if and only if for each $B \in I^Y$, $f^{-1}(intB) \subset intf^{-1}(B)$

Proposition 2.2. Let $f: X \to Y$ and $g: Y \to Z$ be mappings. If f, g are F-open (or F-closed), then $g \circ f$ is F-open(or F-closed).

Proposition 2.3. Let $f: X \to Y$ be injective. Then the inverse mapping $f^{-1}: f(X) \to X$ exists. Moreover f is F-continuous if and only if f^{-1} is F-open.

Proposition 2.4. A mapping $f: X \to Y$ is F-open if and only if for each $A \in I^X$, $f(intA) \subset intf(A)$

Proposition 2.5. Let $f: X \to Y$ be injective. If f is F-continuous, then $intf(A) \subset f(intA)$ for each $A \in I^X$.

Corollary 2.5. Let $f: X \to Y$ be F-continuous, F-open and injective. Then for each $A \in I^X$, $f(intA) = intf(A)$

Proposition 2.6. A mapping $f: X \to Y$ is F-closed if and only if for each $A \in I^X$, $clf(A) \subset f(clA)$

Corollary 2.6. A mapping $f: X \to Y$ is F-continuous and F-closed if and only if for each $A \in I^X$, $f(clA) = clf(A)$

Proposition 2.7. Let $f: X \to Y$ be a mapping.

(1) If X is a fuzzy discrete space, then f is F-continuous.

(2) If Y is a fuzzy indiscrete space, then f is F-continuous.

(3) If X and Y are fuzzy discrete spaces, then f is F-continuous and F-open.

(4) For fuzzy discrete spaces, f is a F-homeomorphism if and only if f is bijective.

Theorem 2.8. Let X and Y be fts's and let $f: X \to Y$ bijective. Then the following are equivalent:

(a) f is a F-homeomorphism.

(b) f and f^{-1} are F-open.

(c) f and f^{-1} are F-closed.

Theorem 2.9. Let X and Y be fts's and let $f: X \to Y$ bijective. Then the following are equivalent:

(a) f^{-1} is F-continuous.

(b) f is F-open.

(c) f is F-closed.

Corollary 2.9. Let X and Y be fts's and let $f: X \to Y$ bijective. Then the following are equivalent:

(a) f is a F-homeomorphism.

(b) f is F-open and F-continuous.

(c) f is F-closed and F-continuous.

Proposition 2.10. For fts's X and Y, let us define $X \cong Y$ to mean that there a F-homeomorphism $f: X \to Y$ Then $\cong$ is an equivalence relation on the family of all fts's.

Theorem 2.11. Let (X, T_X) be a fts, let Y a set and let $f: X \to Y$ a mapping. Let $T_Y = \{U \in I^Y : f^{-1}(U) \in T_X\}$ Then we have the following properties:

(a) T_Y is a fuzzy topology on Y.

(b) $f: X \to Y$ is F-continuous.

(c) If $\mathcal{U}$ is a fuzzy topology on Y such that $f: X \to (Y, \mathcal{U})$ is F-continuous, then T_Y is finer than $\mathcal{U}$.

Theorem 2.12. Let X be a set, let (Y, T_Y) a fts and let $f: X \to Y$ a mapping. Let $T_X = \{U \in I^X$: there is $V \in T_Y$ such that $U = f^{-1}(V).\}$ Then we have the following properties:

(a) T_X is a fuzzy topology on X.

(b) $f: X \to Y$ is F-closed.

(c) If $\mathcal{U}$ is a fuzzy topology on X such that $f: (X, \mathcal{U}) \to Y$ is F-continuous, then T_X is coarser than $\mathcal{U}$.

3. F-continuities of fuzzy set-valued mappings

Definition 3.1. A mapping is said to be *fuzzy set-valued* if it's values are fuzzy sets.

Hence, for instance, $f: I^X \to I^Y$ and $f^{-1}: I^Y \to I^X$ are fuzzy set-valued(See Definition 1.1).

In particular, a fuzzy set-valued mapping $F: Y \to I^X$ is called a *fuzzy multiplication* and denoted by $F: Y \to X$ (See [4]).

Definition 3.2. Let $F: Y \to I^X$ be a fuzzy set-valued mapping, let $A \in I^Y$ and let $\mathcal{B} \subset I^X$. Then :

(1) The *image* of A *under* F, $F(A)$ is defined by :

$F(A) = \{B \in I^X : B = F(a)$ for some $a \in A_0\}$.

(2) The *inverse image* of $\mathcal{B}$ *under* F, $F^{-1}(\mathcal{B})$ is defined by :

$$F^{-1}(\mathcal{B}) = \{y \in Y : F(y) \in \mathcal{B}\}$$

Example 3.3. Let $Y = \{a, b, c\}$, $X = \{x, y, z\}$ and let $F: Y \to I^X$ be the fuzzy set-valued mapping defined by :

$F(a) = \{(x, 0.3), (y, 0.6), (z, 0.7)\}$

$F(b) = \{(x, 0.7), (y, 0.2), (z, 0.6)\}$,

$F(a) = \{(x, 0.4), (y, 0.5), (z, 0.7)\}$.

Let $A = \{(a, 0.4), (b, 0.3), (c, 0,5)\}$, $B = \{(a, 0.3), (b, 0.1), (c, 0.3)\}$ and let $\mathcal{B} = \{\{(x, 0.3), (y, 0.6), (z, 0.7)\}$ $\{(x, 0.7), (y, 0.2), (z, 0.7)\}$ $\{(x, 0.4), (y, 0.6), (z, 0.4)\}\}$ Then :

$F(A) = F(B) = \{F(a), F(b), F(c)\}$

$F(b_{0.5}) = \{F(b)\}$ $F^{-1}(\mathcal{B}) = \{a, b\}$.

For each $A \in I^X$, let I^A denoted the set of all fuzzy sets in X contained in A.

Hence $I^A=\{E\in I^X: E\subset A\}$.

Definition 3.4. Let F_1, F_2: $Y\to I^X$ be fuzzy set valued mappings. Then :

(1) $F_1\subset F_2$ if and only if $F_1(y)\subset F_2(y)$ for each $y\in Y$.

(2) $F=F_1\cup F_2$ if and only if $F(y)=F_1(y)\cup F_2(y)$ for each $y\in Y$.

(3) $F=F_1\cap F_2$ if and only if $F(y)=F_1(y)\cap F_2(y)$ for each $y\in Y$.

From Definition 3.2. and 3.4, we obtain the following result :

Proposition 3.5. Let F_1, F_2: $Y\to I^X$ be fuzzy set-valued and $A\in I^X$. Then :

(1) If $F_1\subset F_2$, then $F_2^{-1}(I^A)\subset F_1^{-1}(I^A)$

(2) If $F=F_1\cup F_2$ then

$$F^{-1}(I^A)=F_1^{-1}(I^A)\cap F_2^{-1}(I^A)$$

(2a) If F_α: $Y\to I^X$ is fuzzy set-valued for each $\alpha\in\Lambda$, then

$$(\cup_{\alpha\in\Lambda}F_\alpha)^{-1}(I^A)=\cap_{\alpha\in\Lambda}F_\alpha^{-1}(I^A).$$

(3) If $F=F_1\cap F_2$ then

$$F_1^{-1}(I^A)\cup F_2^{-1}(I^A)\subset F^{-1}(I^A)$$

(3a) If F_α: $Y\to I^X$ is fuzzy set-valued for each $\alpha\in\Lambda$, then

$$\cup_{\alpha\in\Lambda}F_\alpha^{-1}(I^A)\subset(\cap_{\alpha\in\Lambda}F_\alpha)^{-1}(I^A)$$

Proposition 3.6. Let Y be fts, I_0^X a fuzzy hyperspace, F: $Y\to I_0^X$ a fuzzy set valued mapping and $A\in I^X$. Then :

$$F^{-1}(I_0^A)=\{y\in Y: F(y)\subset A\}$$

and

$$Y-F^{-1}(I_0^{A^c})=\{y\in Y: F(y)qA\}$$

In this case, $F^{-1}(I_0^A)$(resp. $Y-F^{-1}(I_0^{A^c})$) is called the *upper inverse* (resp. *lower inverse*) of A and denoted by $F^+(A)$(resp. $F^-(A)$)(See [3]).

We obtain the similar result as Theorem 2.3 in [3] :

Proposition 3.7. Let Y be a fts, I_0^X a fuzzy hyperspace and F: $Y\to I_0^X$ a fuzzy set-valued mapping. Then

$$F^-(G^c)=Y-F^+(G)\text{ for each } G\in I^X.$$

Theorem 3.8. Let Y be a fts, I_0^X a fuzzy hyperspace and F: $Y\to I_0^X$ a fuzzy set-valued mapping. Then the following are equivalent :

(1) F is F-continuous.

(2) $F^+(A)\in FO(Y)$ for each $A\in FO(X)$ and $F^+(F)\in FC(Y)$ for each $F\in FC(X)$.

(3) $F^-(F)\in FC(Y)$ for each $F\in FC(X)$ and $F^-(A)\in FO(Y)$ for each $A\in FO(X)$.

Corollary 3.8. F is F-continuous at $y_0\in Y$ if and only if the following both implications hold :

$y_0\in F^+(G)\Rightarrow y_0\in intF^+(G)$ whenever G is a fuzzy open set in X,

and

$y_0\in clF^+(K)\Rightarrow y_0\in F^+(K)$ whenever K is a fuzzy closed set in X.

Proposition 3.9. The union of two F-continuous fuzzy set-valued mappings $F=F_0\cup F_1$ is F-continuous.

Corollary 3.9. The union $K\cup L$, considered as a mapping of $I_0^X\times I_0^X$ onto I_0^X is F- continuous.

4. Fuzzy semi-continuities of fuzzy set-valued mappings

Definition 4.1. A fuzzy set-valued mapping F: $Y\to I_0^X$ is said to be :

(1) f*uzzy upper semi-continuous* (in short, *F-usc*) if for each $V\in FO(X)$, $F^+(V)\in FO(Y)$.

(1a) *fuzzy upper semi-continuous* at $y\in Y$ (in short, F-usc at y) if for each $V\in FO(X)$ with $y\in F+(V)$ $y\in intF^+(V)$.

(2) *fuzzy lower semi-continuous* (in short, *F-lsc*) if for each $G\in FC(X)$, $F^+(G)\in FC(Y)$.

(2a) *fuzzy lower semi-continuous* at $y\in Y$ (in short, *F-lsc* at y) if for each $G\in FC(X)$ with $y\in clF^+(G)$, $y\in F^+(G)$.

Proposition 4.2. Let $F:Y\to I_0^X$ be fuzzy set-valued. Then :

(1) F is F-usc if and only if for each $G\in FC(X)$, $F^-(G)\in FC(Y)$

(2) F is F-lsc if and only if for each $V\in FO(X)$, $F^-(V)\in FO(Y)$

The nexted result follows easily from Definition 4.1 :

Proposition 4.3. F is F-usc(resp. F-lsc) if and only if it is F-usc(resp. F-lsc) at each $y \in Y$.

Theorem 4.4. F is F-continuous at y if and only if F is both F-usc and F-lsc at y.
Hence F is F-continuous if and only if F is both F-usc and F-lsc.

Theorem 4.5(Generalized Heine Condition). $F: Y \to I_0^X$ is F-lsc at $y \in Y$ if and only if for each $B \in I^Y$ with $y \in clB$, $F(y) \subset cl[\cup F(B)]$.

Corollary 4.5. F is F-lsc if and only if $\cup F(clB) \subset cl[\cup F(B)]$ for each $B \in I^Y$.

Theorem 4.6 (Generalized Cauchy Condition). $F: Y \to I_0^X$ is F-usc at $y \in Y$ if and only if for each $G \in FO(X)$ with $F(y) \subset G$, there exists $H \in FO(Y)$ such that $y \in H$ and $\cup F(H) \subset G$, i.e., $z_\lambda \in H \Rightarrow F(z) \subset G$.

Theorem 4.7. Let X be a $FT_1(i)$-space and let A a lower fuzzy set in X. If $F: Y \to I_0^X$ is
F-usc, then $\{y \in Y : A \subset F(y)\} \in FC(Y)$.

Theorem 4.8. Let $f: Y \to X$ be a mapping and let X a $FT_1(i)$-space. Let $F: Y \to I_0^X$ be the mapping defined by $F(y) = \{f(y_\lambda)\}$ for each $y \in Y$ and a fixed $\lambda \in I_0$. If F is either F-usc or F-lsc at $y \in Y$, then F-continuous at y_λ.

Theorem 4.9. Let $f: X \to Y$ be a mapping, let $F: Y \to I_0^X$ a fuzzy set-valued mapping and let $f(x_\lambda) = y_\lambda$ If f is F-continuous at x_λ and F is F-usc(resp. F-lsc) at y, then $H = F \circ f$ is F-usc(resp. F-lsc) at x.

Theorem 4.10. The union of two F-usc mappings at y is F-usc at y.

Theorem 4.11. The union of two F-lsc mappings at y is F-lsc at y.
More generally, if each F_t($t \in \Lambda$ arbitrary) is F-lsc at y, then $F = \cup_{t \in \Lambda} F_t$ is F-lsc at y.

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