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타입-2 퍼지 가중치 그래프에서의 최단경로문제

Shortest Path Problem in a Type-2 Fuzzy Weighted Graph

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ABSTRACT

Constructing a shortest path on a graph is a fundamental problem in the area of graph theory. In an application where we cannot exactly determine the weights of edges, fuzzy weights can be used instead of crisp weights, and Type-2 fuzzy weights will be more suitable if this uncertainty varies under some conditions. In this paper, shortest path problem in type-1 fuzzy weighted graphs is extended for type-2 fuzzy weighted graphes. A solution is also given based on possibility theory and extension principle.

Keywords : Shortest Path Problem, Type-2 Fuzzy Weight, Type-2 Fuzzy Comparison

Ⅰ. Introduction

Constructing a shortest path on a graph has been an essential and fundamental problem in the area of graph theory and has lots of interesting applications. In general, crisp values are widely used as the weights on edges, but there are many cases when we cannot determine these weights precisely. In these cases, we can use fuzzy weights instead of crisp weights to express the uncertainty.

If the weight of an edge can vary under some conditions such as time, however, type-1 fuzzy weights are not good enough. For this situation, we need type-2 fuzzy weights instead of type-1 fuzzy weights to show the variations.

This paper handles the shortest path problem in a type-2 fuzzy weighted graph. In section Ⅱ, the shortest problem in a fuzzy weighted graph is briefly explained. This problem will be extended to type-2 fuzzy weighted case in section Ⅲ. A solution for type-2 fuzzy weighted graph is also given in the same section. It is an approach based on possibility theory and extension principle. Conclusions and further works can be found in Section Ⅳ.

Ⅱ. Related Works

In this section, shortest path problem in type-1 fuzzy weighted graph is briefly reviewed.

Shortest path problem in fuzzy weighted graph was first analyzed by Dubois and Prade [5]. There were many researches using different approaches, but we will mainly focus on an approach based on possibility theory proposed by Okada [1].

In case of a graph with crisp weights, we can determine the unique shortest path. But in case of a graph with fuzzy weights, this becomes a nondeterministic problem. Fig 1 shows an example of shortest path problem in a fuzzy weighted graph. There are three verticies, v_1, v_2, and v_3, and three edges, e_{12},

e_{23}, and e_{13}. Each arc has a fuzzy number as its weight. The weight of edge e_{ij} is denoted as $\tilde{l}_{ij}$. We want to calculate the shortest path from v_1 to v_3.

There are two paths from v_1 to v_3. We will call the first path as p_1 and the second path as p_2. To get the shortest path, we must compare the weights (or lengths) of these two paths. The weights of p_1 can be expressed as $\tilde{l}_{12} \oplus \tilde{l}_{23}$ that means the addition of $\tilde{l}_{12}$ and $\tilde{l}_{23}$.

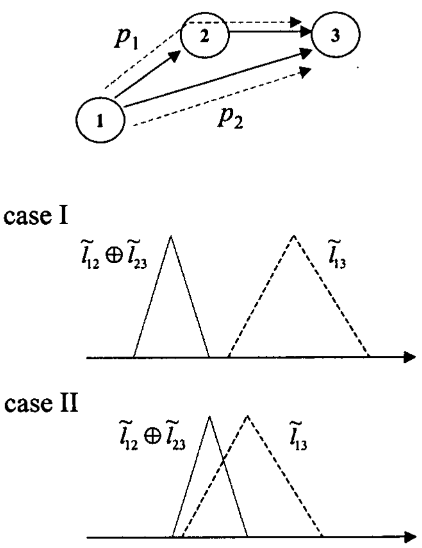

Fig 1 Shortest path problem in a fuzzy weighted graph

Two different cases are shown in Fig 1. The normal line shows the weight of p_1 and the dotted line shows the weight of p_2. In case I, it is obious that p_1 is the shortest path, because the actual value of $\tilde{l}_{23} \oplus \tilde{l}_{34}$ is always smaller than that of $\tilde{l}_{24}$. In case II, however, although p_1 looks relatively shorter than p_2, we cannot guarantee that p_1 is the shortest path because it is also possible that p_2 is shorter than p_1.

Therefore, rather than calculating the exact path, we will calcuate the possibility for each edge to be included in the shortest path. This is the possibility theory based approach. More specific procedure can be defined like following [1].

Definition. Based on the possibility theory, the degree of possibility $D_{ij} \in [0, 1]$ for an edge e_{ij} is defined as

$$D_{ij} = \max_{p \in P_{(i,j)}} \{D_p\}$$

where $P_{(i,\ j)}$ means the set of all paths traversing edge e_{ij}. Moreover, the degree of poossibility D_p for a path p is defined as

$$D_p = \min_{p' \in P_{1n}} \mathrm{Poss}\Big(\sum_{(i,j)\in p} l_{ij} \le \sum_{(i,j)\in p'} l_{ij}\Big)$$

where p' is each path in a set of paths P_{1n} which consists of all paths from the vertex v_1 to v_n, and l_{ij} is a possibility variable of fuzzy edge length a$\tilde{l}_{ij}$long the edge e_{ij}.

D_{ij} means the possibility that edge e_{ij} will be included in the shortest path. After calculating the possibilities D_{ij} for all edges, we can reconstruct the graph using these possibilities. Although the reconstructed graph may not show the exact shortest path, it can show the meaningful candidates with each possibility level. A numerical example of reconstructed network can be found in [1].

III. Type-2 Fuzzy Weighted Graph

We use fuzzy weights to handle the uncertainty problem. But there are some examples where type-1 fuzzy weights are also insufficient. The weights of an edge may vary with time. Or if we assign the weights according to the opinions of expers, and two or more of them may give us different opinions for the same edge. In these examples, type-2 fuzzy weights can be used instead of type-1 fuzzy weights.

In this section, we will extend this approach into the case where type-2 fuzzy weights are assigned on edges. Because we will use the possibility based approach, the general framework that is explained in the previous section will not be changed. The objective of the proposed method is also to calculate D_{ij} values for all edges where the weights are in the form of type-2 fuzzy set, and we will define two operations on type-2 fuzzy weights, addition and comparison, that are both necessary to calculate and compare the length of a path.

For simplicity, only discrete type-2 fuzzy weights will be considered in the proposed method. And there is no restriction such as normalization or convexness on the form of type-2 fuzzy weights. Any kind of discrete type-2 fuzzy sets can be used as type-2 fuzzy weights in the proposed method.

3.1 Addition of type-2 fuzzy weights

Addition operator for type-2 fuzzy weights is constructed based on extension principle.

Definition. Let X be a Cartesian product of universes $X = X_1 \times X_2 \times \ldots \times X_r$, and A_1, A_2, ..., A_r be fuzzy sets in X_1, X_2, ..., X_r, respectively. Let f be a mapping from X to a universe Y such that $y = f(x_1, \ldots, x_r) \in Y$. Zadeh's extension principle allows us to induce from the r fuzzy sets A_i, a fuzzy set B on Y, through f, such that

$$\mu_B(y) = \sup_{x_1,\ldots,x_r : y=f(x_1,\ldots,x_r)} \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r)\}$$
$$\mu_B(y) = 0 \text{ if } f^{-1}(y) = \varnothing$$

where $f^{1}(y)$ is the inverse image of y under f.

Intersection and union operator for type-2 fuzzy sets are used as min and sup operator respectively in the extension principle. The general definition of these two operators can be found in [4].

And the addition operator for type-2 fuzzy weights can be defined like following.

Definition. Let two discrete type-2 fuzzy weights be $\tilde{\tilde{A}} = \sum \tilde{\mu}_{\tilde{\tilde{A}}}(x)/x$ and $\tilde{\tilde{B}} = \sum \tilde{\mu}_{\tilde{\tilde{B}}}(y)/y$ where $\tilde{\mu}_{\tilde{\tilde{A}}}(x) = \sum f_x(u)/u$ and $\tilde{\mu}_{\tilde{\tilde{B}}}(y) = \sum g_y(w)/w$. If we denote the addition of these two type-2 fuzzy weights as $\tilde{\tilde{A}} \oplus \tilde{\tilde{B}}$, its membership function, $\mu_{\tilde{\tilde{A}}\oplus\tilde{\tilde{B}}}(z)$, is defined as

$$\begin{aligned} \tilde{\mu}_{\tilde{\tilde{A}}\oplus\tilde{\tilde{B}}}(z) &= \bigcup_{z=x+y} (\tilde{\mu}_{\tilde{\tilde{A}}}(x) \cap \tilde{\mu}_{\tilde{\tilde{B}}}(y)) \\ &= \bigcup_{z=x+y} ((\sum_i f_x(u_i)/u_i) \cap (\sum_j g_y(w_j)/w_j)) \\ &= \bigcup_{z=x+y} (\sum_{i,j} (f_x(u_i) \wedge g_y(w_j))/(u_i \wedge w_j)) \end{aligned}$$

Example. Let two type-2 fuzzy weights be

$$\begin{aligned} \tilde{\tilde{A}} &= \tilde{\mu}_{\tilde{\tilde{A}}}(2)/2 + \tilde{\mu}_{\tilde{\tilde{A}}}(3)/3 \\ &= (0.5/0 + 0.7/0.1)/2 + (0.6/0.2)/3 \end{aligned}$$

and

$$\begin{aligned} \tilde{\tilde{B}} &= \tilde{\mu}_{\tilde{\tilde{B}}}(3)/3 + \tilde{\mu}_{\tilde{\tilde{B}}}(4)/4 \\ &= (0.3/0.4 + 0.9/0.8)/3 + (1.0/0.1 + 0.5/0.7)/4 \end{aligned}$$

Then we can calculate the addition of these two weights like following.

$$\begin{aligned} \tilde{\tilde{A}} \oplus \tilde{\tilde{B}} &= (\tilde{\mu}_{\tilde{\tilde{A}}}(2) \cap \tilde{\mu}_{\tilde{\tilde{B}}}(3))/5 + \\ &\quad ((\tilde{\mu}_{\tilde{\tilde{A}}}(2) \cap \tilde{\mu}_{\tilde{\tilde{B}}}(4)) \cup (\tilde{\mu}_{\tilde{\tilde{A}}}(3) \cap \tilde{\mu}_{\tilde{\tilde{B}}}(3)))/6 \\ &\quad (\tilde{\mu}_{\tilde{\tilde{A}}}(3) \cap \tilde{\mu}_{\tilde{\tilde{B}}}(4))/7 \\ &= (0.5/0.1 + 0.7/0.1)/5 + (0.6/0.2)/6 + (0.6/0.1 + 0.5/0.2)/7 \end{aligned}$$

Using this addition operator, the summed weights (or length) of a path can be calculated in the case of type-2 fuzzy weighted graph.

3.2 Comparison of type-2 fuzzy weights

After calculating the length of paths, we need to compare them to get the possibility D_p for each path. Because we want to get the possibility for each path to be a shortest one, we must use the comparison operator that is compatible with this approach.

The proposed comparison operator is based on "possibility of appearance". For example, let us consider the following two discrete fuzzy sets.

$$\tilde{A} = 0.5/1 + 1.0/3$$
$$\tilde{B} = 0.3/2 + 0.5/4$$

If we denote the actual value of fuzzy set $\tilde{X}$ as $\mathrm{av}(\tilde{X})$, then $\mathrm{av}(\tilde{A})$ can be 1 or 3. Although we don't know whether $\mathrm{av}(\tilde{A})$ is 1 or 3, we can assume that the possibility of $\mathrm{av}(\tilde{A}) = 3$ is twice greater than the possibility of $\mathrm{av}(\tilde{A}) = 1$. We can make similar assumption on the possibility of $\mathrm{av}(\tilde{B})$.

If $\mathrm{av}(\tilde{A}) = 3$ and $\mathrm{av}(\tilde{B}) = 2$, then $\tilde{A}$ will be greater than $\tilde{B}$. But if $\mathrm{av}(\tilde{A}) = 1$ and $\mathrm{av}(\tilde{B}) = 2$, then $\tilde{B}$ will be greater than $\tilde{A}$. Therefore we cannot exactly determine if $\tilde{A} > \tilde{B}$ or $\tilde{B} > \tilde{A}$. But we can calculate the possibility of

$\text{Poss}(\tilde{A} > \tilde{B})$ using the assumption given above. If we denote the relative possibilities for a situation K happens as $s(K)$ that is called a satisfaction degree, then

$$s(\text{av}(\tilde{A})=1 \wedge \text{av}(\tilde{B})=2) : s(\text{av}(\tilde{A})=1 \wedge \text{av}(\tilde{B})=4) : s(\text{av}(\tilde{A})=3 \wedge \text{av}(\tilde{B})=2) : s(\text{av}(\tilde{A})=3 \wedge \text{av}(\tilde{B})=4) = 3:5:6:10$$

And because this covers all the possible situations, we can say that the possibility that A is greater than B will be

$$\text{Poss}(\tilde{A} > \tilde{B}) = \frac{6}{3+5+6+10} = 0.25$$

The possibility of appearance for type-2 fuzzy sets are proportional to both type-1 and type-2 membership values. Therefore we can get the satisfaction degree for type-2 fuzzy weights can be defined like following [2].

Definition. Let two discrete type-2 fuzzy weights be $\tilde{\tilde{A}} = \sum \tilde{\mu}_{\tilde{\tilde{A}}}(x)/x$ and $\tilde{\tilde{B}} = \sum \tilde{\mu}_{\tilde{\tilde{B}}}(y)/y$ where $\tilde{\mu}_{\tilde{\tilde{A}}}(x) = \sum f_x(u)/u$ and $\tilde{\mu}_{\tilde{\tilde{B}}}(y) = \sum g_y(w)/w$. The satisfaction degree for $\text{av}(\tilde{\tilde{A}}) = x$ and $\text{av}(\tilde{\tilde{B}}) = y$ can be defined as

$$s(\text{av}(\tilde{\tilde{A}}) = x \wedge \text{av}(\tilde{\tilde{B}}) = y) = \sum\sum u \cdot f(u) \cdot w \cdot g(w)$$

And the possibility that $\tilde{\tilde{A}} < \tilde{\tilde{B}}$ is defined as

$$\text{Poss}(\tilde{\tilde{A}} < \tilde{\tilde{B}}) = \frac{\sum_{(x,y)\in\Omega(\tilde{\tilde{A}}<\tilde{\tilde{B}})} s(\text{av}(\tilde{\tilde{A}}) = x \wedge \text{av}(\tilde{\tilde{B}}) = y)}{\sum_{(x,y)\in\Omega(T)} s(\text{av}(\tilde{\tilde{A}}) = x \wedge \text{av}(\tilde{\tilde{B}}) = y)}$$

where $\Omega(\tilde{\tilde{A}} < \tilde{\tilde{B}})$ is the set of all the possible situations where $\text{av}(\tilde{\tilde{A}}) < \text{av}(\tilde{\tilde{B}})$, and $\Omega(T) = \Omega(\tilde{\tilde{A}} < \tilde{\tilde{B}}) + \Omega(\tilde{\tilde{A}} = \tilde{\tilde{B}}) + \Omega(\tilde{\tilde{A}} > \tilde{\tilde{B}})$. The possibility of $\tilde{\tilde{A}} = \tilde{\tilde{B}}$ and $\tilde{\tilde{A}} > \tilde{\tilde{B}}$ can be calculated using similar method.

Using this comparison operator, the lengths of two paths can be compared and we can get the possibility that p_i will be shorter than p_j. The procedure to reconstruct the shortest path is the same as in the case of type-1 fuzzy weighted graph.

Ⅳ. Conclusion

The shortest path problem in a type-2 fuzzy weighted graph is explained in this paper. And we propose a method based on possibility theory for type-2 fuzzy weighted graph. This method uses extension principle and possibility theory to add and compare type-2 fuzzy weights.

Currently the most serious problem of possibility-based approach is the computational complexity. Because we must compare all the possible combinations of paths in a graph to get a result, this is a problem of NP class even in the case of type-1 fuzzy weighted graph [1]. If the number of vertices grows larger, it will take much time to get the result graph.

Therefore a study on an approximation algorithm based on heuristics is considered as a future research direction.

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Ⅳ. Reference

[1] Shinkoh Okada, "Interactions among paths in fuzzy shortest path problems," *Proc. of the Joint 9th IFSA World Congress and 20th NAFIPS International Conference*, pp. 41-46, 2001.

[2] Seungsoo Lee and Kwang H. Lee, "Comparison of type-2 fuzzy sets with satisfaction function," *Proc. of the 2nd International Symposium on Advanced Intelligent Systems*, pp. 436-439, 2001.

[3] Keon-Myung Lee, Choong-Ho Cho, and Hyung Lee-Kwang, "Ranking Fuzzy Values with Satisfaction Function," *Fuzzy Sets and Systems*, vol. 64, pp. 295-309, 1994.

[4] Nilesh N. Karnik and Jerry M. Mendel, "Introduction to type-2 fuzzy logic systems," *USC Report*, 1998.

[5] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, 1980.