Generating Chaos from Discrete TS Fuzzy Systems*

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Abstract

In this paper, a simple and systematic control design method is proposed for a discrete-time Takagi-Sugeno (TS) fuzzy system, which employs the parallel distributed compensation (PDC) to determine the structure of a fuzzy controller so as to make all the Lyapunov exponents of the controlled TS fuzzy system strictly positive. This approach is proven to be mathematically rigorous for anticontrol of chaos for a TS fuzzy system, in the sense that any given discrete-time TS fuzzy system can be made chaotic by the designed PDC controller along with the mod-operation. A numerical example is included to visualize the anticontrol effect.

Keywords:
Anticontrol of chaos; TS fuzzy system; Parallel distributed compensation (PDC); Lyapunov exponents

Introduction

Today, it is well known that most conventional control methods and many special techniques can be used for chaos control [1], for which no matter the purpose is to reduce "bad" chaos or to introduce "good" chaos, numerous control methodologies have been proposed, developed, tested, and applied. Similar to conventional systems control, the concept of "controlling chaos" is first to mean ordering or suppressing chaos in the sense of stabilizing chaotic system responses. In this pursuit, numerical and experimental simulations have convincingly demonstrated that chaotic systems respond well to these control strategies. These methods of ordering chaos include the now-familiar OGY method [2,3], feedback controls [4,5], and fuzzy control [6-8], etc., to name just a few.
However, controlling chaos has also encompassed many nontraditional tasks, particularly those of enhancing or generating chaos when it is beneficial. The process of chaos control is now understood as a transition between chaos and order, and sometimes from order to chaos, depending on the application of interest. The task of purposely creating chaos, sometimes called chaotification or anticontrol of chaos, has attracted increasing attention in recent years due to its great potential in nontraditional applications such as those found within the context of physical, chemical, mechanical, electrical, optical, and particularly biological and medical systems [9-11]. Recently, there have been some successful reports on anticontrolling chaos [10-12]. Although these reports are essentially experimental or semi-analytical, in the sense that no explicit and quantitative computational formulas are provided with rigorous mathematical justification, they are nevertheless interesting and promising. Two simple yet mathematically rigorous control methods from the engineering feedback control approach were developed [11-13], where a positive state-feedback controller with an uniformly bounded control-gain sequence can be designed to make all Lyapunov exponents of the controlled system strictly positive, or arbitrarily assigned (with any positive, zero and negative arrangements). Moreover, such a controller can be designed for an arbitrarily given, n-dimensional dynamical system that could be originally nonchaotic or even asymptotically stable.
The possible interactions between fuzzy logic and chaos theory have been explored since the 1990s. The explorations have been carried mainly on fuzzy modeling of chaotic systems using the Takagi-Sugeno (TS) model [6-8,16], linguistic descriptions [17] and fuzzy control of chaos via an LMI-based fuzzy control system design [7,8]. In these investigations, to design a fuzzy controller chaotic systems are represented by TS fuzzy models, and then the LMI-based design problems are defined and employed to find feedback gains of the fuzzy controllers that can satisfy some specifications such as stability, decay rate, and constraints on the control input and output of the overall fuzzy control systems.

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In this paper, the problem of anticontrolling chaos in TS fuzzy systems is studied. The goal is to make nonchaotic or even stable TS fuzzy systems chaotic. The concept of PDC is utilized to design a fuzzy controller which can make all the Lyapunov exponents strictly positive, for any given n-dimensional discrete-time TS fuzzy system that could be originally nonchaotic or even asymptotically stable. Actually, these Lyapunov exponents can also be rearranged by the PDC in such a way that one has any desired (positive, zero, and/or negative) Lyapunov exponents in an arbitrary order.

Chaotic TS Fuzzy Model

The TS fuzzy model [7,8,16] is captured by a set of fuzzy implications, which characterize local relations of the system in the state space. The main feature of a TS model is to express the local dynamics of each fuzzy implication (rule) by a linear state-space system model. The overall fuzzy system is then modeled by fuzzy "blending" of these local linear system models.

Specifically, a general TS fuzzy system is described as follows:

Rule i: IF $x_1(k)$ is M_{i1} ...and $x_n(k)$ is M_{in}

$$\text{THEN } \mathbf{x}(k+1) = \mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{u}(k), \quad (1)$$

where

$$\mathbf{x}^T(k) = [x_1(k), x_2(k), \Lambda\ x_n(k)],$$

$$\mathbf{u}^T(k) = [u_1(k), u_2(k), \Lambda\ u_m(k)],$$

$i = 1,2,\Lambda\ ,r,$ in which r is the number of IF-THEN rules, M_{ij} are fuzzy sets, and the equation $\mathbf{x}(k+1) = \mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{u}(k)$ is the output from the ith IF-THEN rule.

Assume that $\mathbf{A}_i$ and $\mathbf{B}_i$, $i = 1,2,\Lambda\ ,r$, are uniformly bounded; that is, there are constants $N\ and\ Q$ such that

$$\sup_{1\le i\le r}\|\mathbf{A}_i\| \le N < \infty \quad and \quad \sup_{1\le i\le r}\|\mathbf{B}_i\| \le Q < \infty$$

with

$$\det(\mathbf{B}_i) \ge \eta > 0 \quad \text{for all } i = 1,\ldots,r,$$

where $\|\cdot\|$ denote the spectral norm of a finite-dimensional matrix, that is, the largest singular value of the matrix.

Now, given a pair of $(\mathbf{x}(k), \mathbf{u}(k))$, the final output of the fuzzy system is inferred as follows:

$$\mathbf{x}(k+1) = \frac{\sum_{i=1}^{r} w_i(k)\{\mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{u}(k)\}}{\sum_{i=1}^{r} w_i(k)}, \quad (2)$$

where

$$w_i(k) = \prod_{j=1}^{n} M_{ij}(x_j(k)),$$

$M_{ij}(x_j(k))$ is the degree of membership of $x_j(k)$ in M_{ij}, with

$$\begin{cases} \sum_{i=1}^{r} w_i(k) > 0 \\ w_i(k) \ge 0, \end{cases} \qquad i = 1,2,\Lambda\ r$$

By introducing $h_i(k)(= w_i(k)/\sum_{i=1}^{r} w_i(k))$ instead of $w_i(k)$, (2) is rewritten as

$$\mathbf{x}(k+1) = \sum_{i=1}^{r} h_i(k)\{\mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{u}(k)\}. \quad (3)$$

Note that

$$\begin{cases} \sum_{i=1}^{r} h_i(k) = 1 \\ h_i(k) \ge 0, \end{cases} , \quad i = 1,2,\Lambda\ r \quad (4)$$

in which $h_i(k)$ can be regarded as the normalized weight of the IF-THEN rules.

Definition 1 (Chaotic TS fuzzy model): TS fuzzy model (3) is said to be *chaotic in the sense of Li and Yorke* if it has a snap-back repeller. Thus, it can be called a *chaotic TS fuzzy model*.

Anticontrol of Chaos via PDC

The parallel-distributed compensation (PDC) is employed here to determine a structure of a fuzzy controller from a given TS fuzzy model. Each control rule is constructed from the corresponding rule of the TS fuzzy model in the PDC. The designed fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts. The PDC provides the following fuzzy control rule structure from the fuzzy model (1):

Control Rule i:

IF $x_1(k)$ is M_{i1} ...and $x_n(k)$ is M_{in},

$$\text{Then } \mathbf{u}(k) = -\mathbf{F}_i\mathbf{x}(k), \quad i = 1,2,\Lambda\ r. \quad (5)$$

The fuzzy control rules have linear state feedback laws in the consequent parts. The overall fuzzy controller is represented by

$$\mathbf{u}(t) = -\frac{\sum_{i=1}^{r} w_i(k)\mathbf{F}_i\mathbf{x}(k)}{\sum_{i=1}^{r} w_i(k)} = -\sum_{i=1}^{r} h_i(k)\mathbf{F}_i\mathbf{x}(k). \quad (6)$$

To be practical, the control-gain matrices $\{\mathbf{F}_i\}_{i=1}^{r}$ should be uniformly bounded:

$$\sup_{1\le i\le r}\|\mathbf{F}_i\| \le M < \infty \quad (7)$$

for some constant M.
By substituting (6) into (3), we obtain

$$\mathbf{x}(k)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(k)h_j(k)\{\mathbf{A}_i-\mathbf{B}_i\mathbf{F}_j\}\mathbf{x}(k). \quad (8)$$

System (8) can be also written as

$$\begin{aligned}
&\mathbf{x}(k+1)\\
&=\sum_{i=1}^{r}h_i(k)h_i(k)\{\mathbf{A}_i-\mathbf{B}_i\mathbf{F}_i\}\mathbf{x}(k)\\
&\quad+2\sum_{i<j}^{r}h_i(k)h_j(k)\frac{\{\mathbf{A}_i-\mathbf{B}_i\mathbf{F}_j\}+\{\mathbf{A}_j-\mathbf{B}_j\mathbf{F}_i\}}{2}\mathbf{x}(k)\\
&=\sum_{i=1}^{r}h_i(k)h_i(k)\mathbf{G}_{ii}\mathbf{x}(k)\\
&\quad+2\sum_{i<j}^{r}h_i(k)h_j(k)\left\{\frac{\mathbf{G}_{ij}+\mathbf{G}_{ji}}{2}\right\}\mathbf{x}(k)
\end{aligned} \quad (9)$$

where $\mathbf{G}_{ij}=\mathbf{A}_i-\mathbf{B}_i\mathbf{F}_j$ for all $i,j=1,\ldots,r$.

We choose $\mathbf{B}_i=\mathbf{B}, i=1,2,\Lambda,r$, so that (9) can be rewritten as:

$$\begin{aligned}
\mathbf{x}(k+1)&=\sum_{i=1}^{r}h_i(k)\{\mathbf{A}_i-\mathbf{B}\mathbf{F}_i\}\mathbf{x}(k)\\
&=\sum_{i=1}^{r}h_i(k)\mathbf{G}_{ii}\mathbf{x}(k)
\end{aligned} \quad (10)$$

Theorem 1[8]: The TS fuzzy system (3) is exactly linearizable via the PDC fuzzy controller (5) if there exists feedback gains $\mathbf{F}_i$ such that

$$\begin{aligned}
&\{(\mathbf{A}_1-\mathbf{B}\mathbf{F}_1)-(\mathbf{A}_i-\mathbf{B}\mathbf{F}_i)\}^T\times\\
&\{(\mathbf{A}_1-\mathbf{B}\mathbf{F}_1)-(\mathbf{A}_i-\mathbf{B}\mathbf{F}_i)\}=0
\end{aligned} \quad (11)$$

for $i=2,3,\Lambda,r$. The overall control system is linearized as

$$\mathbf{x}(k+1)=\mathbf{G}\mathbf{x}(k) \quad (12)$$

where $\mathbf{G}=\mathbf{G}_{ii},\ i=2,3,\ldots,r.$

It is well known that the *j*th Lyapunov exponent of the orbit $\{\mathbf{x}_k\}_{k=0}^{\infty}$ of the controlled system (12), starting from the given $\mathbf{x}_0$, is defined by

$$\lambda_j=\lim_{k\to\infty}\frac{1}{k}\ln\left|\mu_j(\mathbf{G}^k)\right|,\quad j=1,2,\Lambda\ n, \quad (13)$$

where $\mu_j(\mathbf{G}^k)$ is the *j*th singular value of matrix $\mathbf{G}^k$.
In the controlled system (12), we want to design the constant matrices $\{\mathbf{F}_i\}_{i=1}^{r}$, given in (6), such that the Lyapunov exponents of the controlled system orbit $\{\mathbf{x}_k\}_{k=0}^{\infty}$ can be arbitrarily assigned:

$$\lambda_j(x_0)=\sigma_j,\quad j=1,2,\Lambda,n, \quad (14)$$

where $\{\sigma_j\}_{j=1}^{n}$ are arbitrarily chosen by the user, which may be positive, zero, or negative (but all finite).
A convenient choice is, simply,

$$\mathbf{G}=diag\{e^{\sigma_1},e^{\sigma_2},\mathrm{K},e^{\sigma_n}\}.$$

It is clear that the eigenvalues of $\mathbf{G}$ are all larger than 1 if $\sigma_j>0,\ j=1,2,\Lambda,n$.

Matrices $\mathbf{F}_i,\ i=1,2,\Lambda,r$, can then be obtained and they are uniformly bounded.

Theorem 2: The resulting overall controlled system (12), along with the mod-operation:

$$\mathbf{x}(k+1)=\mathbf{G}\mathbf{x}(k) \qquad (\text{mod-1}) \quad (15)$$

where $\mathbf{G}=diag\{e^{\sigma_1},e^{\sigma_2},\mathrm{K},e^{\sigma_n}$ and $\sigma_i>0$, $i=1,2,\Lambda,n$, is *chaotic in the sense of Li and Yorke.*

A Simulation Example

Consider a nonchaotic discrete-time TS fuzzy model given as follows:

Rule 1: IF $x(t)$ is M_1,

$$\text{THEN}\ \begin{bmatrix}x(t+1)\\y(t+1)\end{bmatrix}=\mathbf{A}_1\begin{bmatrix}x(t)\\y(t)\end{bmatrix}+\mathbf{Bu}$$

Rule 2: IF $x(t)$ is M_2,

$$\text{THEN}\ \begin{bmatrix}x(t+1)\\y(t+1)\end{bmatrix}=\mathbf{A}_2\begin{bmatrix}x(t)\\y(t)\end{bmatrix}+\mathbf{Bu}$$

where

$$\mathbf{A}_1=\begin{bmatrix}d&0.3\\1&0\end{bmatrix},\quad \mathbf{A}_2=\begin{bmatrix}-d&0.3\\1&0\end{bmatrix},$$

$x(t)\in[-d,d]$ *and* $d>0$, with membership functions

$$M_1=\frac{1}{2}\left(1-\frac{x(t)}{d}\right),\quad M_2=\frac{1}{2}\left(1+\frac{x(t)}{d}\right).$$

Without control (i.e., $\mathbf{u}=0$), the system is stable as shown in Figure 1.
Using two desired Lyapunov exponents,
$\sigma_1=\ln(1.9)=0.6418539$, and
$\sigma_2=\ln(2.0)=0.6931472$,

For the sake of simplicity, we assume that $\mathbf{B}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $d=2$. We completed the design of the feedback controller by following the procedure described above. We obtained the controlled system output as shown in Figure 2—Figure 4. The output trajectory is displayed in the x_t-x_{t+1} phase plane after some mod-2 operations (they are obviously equivalent to mod-1 operations for anticontrol), which has the above-indicated Lyapunov exponents $\lambda_1 = \sigma_1$ and $\lambda_2 = \sigma_{21}$.

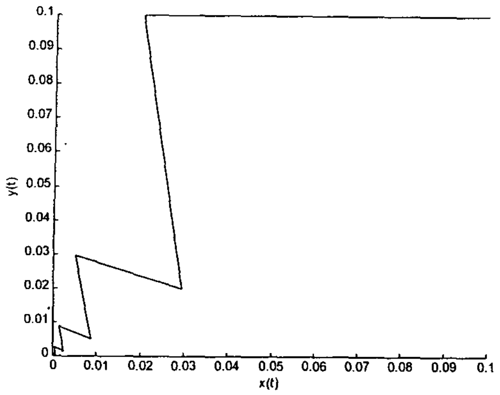

Fig.1-The system orbit without control

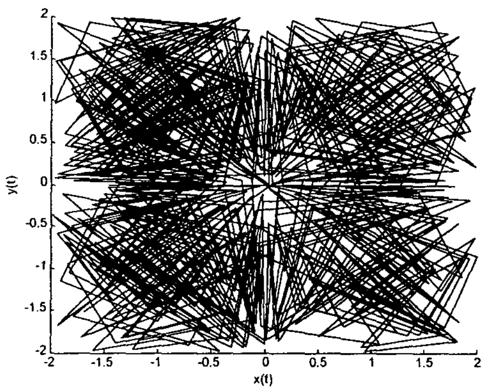

Fig.2-The chaotic orbits of the anticontrol system

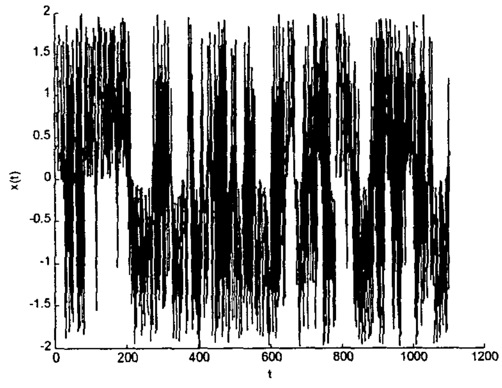

Fig.3-The phase portrait of $t - x(t)$

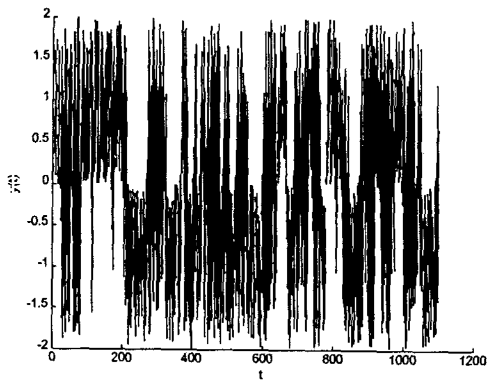

Fig.4-The phase portrait of $t - y(t)$

Conclusions

A simple, yet mathematically precise and rigorous PDC control design procedure has been derived in this paper, which can rearrange all the Lyapunov exponents of the controlled system according to the user's desire, namely, to make them positive, zero, and/or negative in an arbitrary order, for any given n-dimensional discrete-time TS fuzzy system that could be originally nonchaotic or even asymptotically stable.

As is known, many chaotic systems, such as Lorenz and Henon systems, can be represented by TS fuzzy models. Therefore, they can be controlled by using LMI-based fuzzy control system design. Using this proposed PDC control design approach, one can make nonchaotic or even asymptotically stable TS fuzzy systems chaotic, which provides a means to further explore the interaction between fuzzy control and chaos theory, which has great potential in future engineering applications of chaos. To our knowledge, there does not seem to be much research done on generating chaos via fuzzy control systems prior to this work.

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