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On gf. γ-closed sets and g*f. γ-closed sets

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ABSTRACT

Park et al. [Proc. KFIS Fall Conf. 10(2) (2000), 59-62] defined fuzzy γ-open sets by using an operation γ on a fts (X, τ) and investigated the related fuzzy topological properties of the associated fuzzy topology τ_γ and τ. As generalizations of the notion of fuzzy γ-closed sets, we define gf. γ-closed sets and g*f. γ-closed sets and study basic properties of these sets relative to union and intersection. Also, we introduce and study two classes of fts's called fuzzy γ-T_* and fuzzy γ-$T_{1/2}$ spaces by using the notions of gf. γ-closed and g*f. γ-closed sets.

1. Introduction

In 1968, Chang[2] introduced the concept of a fuzzy topological space(in short, fts) by using the fuzzy set. Since then, many authors have contributed to the development of this theory.

Balasubramanian and Sundaram[1] introduced the notion of generalized fuzzy closed sets and study their properties in 1997. Recently, Park et al. [6] defined fuzzy γ-open sets by using an operation γ on a fts (X, τ) and investigated the related fuzzy topological properties of the associated fuzzy topology τ_γ and τ.

As generalizations of the notion of fuzzy γ-closed sets, we define gf. γ-closed sets and g*f. γ-closed sets and study basic properties of these sets relative to union and intersection. Also, we introduce and study two classes of fts's called fuzzy γ-T_* and fuzzy γ-$T_{1/2}$ spaces by using the notions of gf. γ-closed and g*f. γ-closed sets.

An operation γ on the fuzzy topology τ is a mapping from τ into the fuzzy power set I^X of X such that $V \leq V^\gamma$ for each $V \in \tau$, where V^γ denoted the value of γ at V. It is denoted by $\gamma : \tau \to I^X$. The operators defined by $\gamma(V) = \text{Int}(V)$, $\gamma(V) = \text{Cl}(V)$ and $\gamma(V) = \text{Int}(\text{Cl}(V))$ are examples of the operation γ.

Definition 1.1 [6] A subset A of a fts (X, τ) is called fuzzy γ-open in (X, τ) if for each fuzzy point $x_\alpha \in A$, there exists a fuzzy open set U containing x_α such that $U^\gamma \leq A$. τ_γ will denote the set of all fuzzy γ-open sets in (X, τ).

Definition 1.2 [6] Let (X, τ) be a fts. An operation γ is said to be

(a) *regular* if for every fuzzy open neighborhoods (simply, fo-nbd) U and V of each fuzzy point $x_\alpha \in X$, there exists a fo-nbd W of x_α such that $W^\gamma \leq U^\gamma \wedge V^\gamma$;

(b) *open* if for every fo-nbd U of each point $x_\alpha \in X$, there exists a fuzzy γ-open set V such that $x_\alpha \in V$ and $V \leq U^\gamma$.

Proposition 1.3 [6] Let $\gamma: \tau \to I^X$ be a regular operation on τ.

(a) If A and B are fuzzy γ-open, then $A \wedge B$ is

fuzzy γ-open.
(b) τ_γ is a fuzzy topology on X such that $\tau_\gamma \subset \tau$.

Definition 1.4 A fuzzy point x_α of X is in the *fuzzy γ-closure* [6] of fuzzy set A of X, denoted by $\mathrm{Cl}_\gamma(A)$, if $U^\gamma \mathrm{q} A$ for any fo-q-nbd U of x_α. A fuzzy point x_α of X is in the *fuzzy γ-interior* of A, denoted by $\mathrm{Int}_\gamma(A)$, if $U^\gamma \leq A$ for some fo-q-nbd U of x_α .

Proposition 1.5 Let U be a fuzzy open set and let A be any fuzzy set of fts (X, τ). If $A \,\overline{\mathrm{q}}\, U^\gamma$, then $\mathrm{Cl}_\gamma(A) \,\overline{\mathrm{q}}\, U$.

2. gf. γ-closed and g*f. γ-closed sets

In this section, we introduce the notion of gf. γ-closed and g*f. γ-closed sets investigate the relation between them.

Definition 2.1 A fuzzy set A of fts(X, τ) is said to be
(a) *generalized fuzzy γ-closed* (shortly, gf. γ-closed) if $\mathrm{Cl}_\gamma(A) \leq U$ whenever $A \leq U$ and U is fuzzy open in (X, τ).
(b) *generalized* fuzzy γ-closed* (shortly, g*f. γ-closed) if $\mathrm{Cl}_\gamma(A) \leq U$ whenever $A \leq U$ and U is fuzzy γ-open in (X, τ);

Remark 2.2 From above definition and Definition 2.1 of [1], we obtain the following diagram:

fuzzy γ-closed → gf. γ-closed → g*f. γ-closed
↓ ↓
fuzzy closed → gf-closed

Example 2.3 Let $X=\{a,b,c\}$ and $\tau=\{1_X, 0_X, A_1, A_2, A_3, A_4\}$ where

$$A_1(a)=0.2,\ \ A_1(b)=A_1(c)=0.8;$$
$$A_2(a)=A_2(c)=0.8,\ \ A_2(b)=0.2;$$
$$A_3(a)=A_3(b)=0.2,\ \ A_3(c)=0.8;$$
$$A_4(a)=A_4(b)=A_4(c)=0.8.$$

Let $\gamma: \tau \to I^X$ be an operation defined by $A_3^\gamma = A_3$ and $A^\gamma = \mathrm{Cl}(A)$ if $A \neq A_3$. Let B_i $(i=1,2,3,4)$ be the fuzzy sets of X defined as follows:

$$B_1(a)=0.5,\ \ B_1(b)=0.4,\ \ B_1(c)=0.6;$$
$$B_2(a)=0.2,\ \ B_2(b)=0.8,\ \ B_2(c)=0.2;$$
$$B_3(a)=0.2,\ \ B_3(b)=0.9,\ \ B_3(c)=0.2;$$
$$B_4(a)=0.2,\ \ B_4(b)=0.2,\ \ B_4(c)=0.2.$$

Then we have
(a) B_1 is g*f. γ-closed but not gf-closed.
(b) B_2 is fuzzy closed but not fuzzy γ-closed.
(c) B_3 is gf. γ-closed but neither fuzzy γ-closed nor fuzzy closed.
(d) B_4 is fuzzy closed but not g*f. γ-closed.

Theorem 2.4 For fuzzy subsets A, B of a fts X, the following statements are true:
(a) If A and B are g*f. γ-closed, then $A \vee B$ is g*f. γ-closed.
(b) If A and B are gf. γ-closed, then $A \vee B$ is gf. γ-closed.

However, the intersection of two gf. γ-closed (resp. g*f. γ-closed) sets need not gf. γ-closed (resp. g*f. γ-closed).

Example 2.5 Let (X, τ) be a fts given in Example 2.3.
(a) Let A and B be fuzzy sets defined as follows:

$$A(a)=0.1,\ \ A(b)=0.1,\ \ A(c)=0.9\ ;$$
$$B(a)=0.5,\ \ B(b)=0.4,\ \ B(c)=0.6\ .$$

Then A and B are g*f. γ-closed but $A \wedge B$ is not g*f. γ-closed.
(b) Let A and B be fuzzy sets defined as follows:

$$A(a)=0.9,\ \ A(b)=0.2,\ \ A(c)=0.9\ ;$$
$$B(a)=0.2,\ \ B(b)=0.9,\ \ B(c)=0.2$$

Then A and B are gf. γ-closed but $A \wedge B$ is not gf. γ-closed.

Theorem 2.6 Let $\gamma: \tau \to I^X$ be an open operation.
(a) If A is g*f. γ-closed and if $A \leq B \leq \mathrm{Cl}_\gamma(A)$, then B is g*f. γ-closed.
(b) If A is gf. γ-closed and if $A \leq B \leq \mathrm{Cl}_\gamma(A)$,

then B is gf. γ-closed.

Proof. (a): Let U be a fuzzy γ-open set such that $B \leq U$. Since $A \leq U$ and A is g*f. γ-closed, $\mathrm{Cl}_\gamma(A) \leq U$. But $\mathrm{Cl}_\gamma(B) \leq \mathrm{Cl}_\gamma(A)$ since operation γ is open. So $\mathrm{Cl}_\gamma(B) \leq U$ and hence B is g*f. γ-closed.
(b): The proof is similar to (a).

By Remark 2.12 (c) in [6], we have following:

Corollary 2.7 Let $\gamma: \tau \to I^X$ be an open operation.
(a) If A is g*f. γ-closed and if $A \leq B \leq \tau_\gamma\text{-Cl}(A)$ then B is g*f. γ-closed.
(b) If A is gf. γ-closed and if $A \leq B \leq \tau_\gamma\text{-Cl}(A)$ then B is gf. γ-closed. .

3. gf. γ-open and g*f. γ-open sets

In this section we introduce the notions of gf. γ-open and g*f. γ-open sets and study their basic properties.

Definition 3.1 A fuzzy set A of a fts (X, τ) is called g*f. γ-open (resp. gf. γ-open) if the complement $1-A$ is g*f. γ-closed (resp. gf. γ-closed).

Theorem 3.2 (a) A fuzzy set A is g*f. γ-open if and only if $F \leq \mathrm{Int}_\gamma(A)$ whenever F is fuzzy γ-closed and $F \leq A$.
(b) A fuzzy set A is gf. γ-open if and only if $F \leq \mathrm{Int}_\gamma(A)$ whenever F is fuzzy closed and $F \leq A$.

Theorem 3.3 (a) If fuzzy γ is open and if A and B are fuzzy γ-separated (i.e., $\mathrm{Cl}_\gamma(A) \wedge B = 0_X = A \wedge \mathrm{Cl}_\gamma(B)$) g*f. γ-open sets, then $A \vee B$ is g*f. γ-open.
(b) If A and B are fuzzy γ-separated gf. γ-open sets, then $A \vee B$ is gf. γ-open.

Proof. (a): Let F be a fuzzy γ-closed set and $F \leq A \vee B$. Since γ is open, $F \wedge \mathrm{Cl}_\gamma(A)$ is fuzzy γ-closed and $F \wedge \mathrm{Cl}_\gamma(A) \leq A$, and hence by Theorem 3.2 (a), $F \wedge \mathrm{Cl}_\gamma(A) \leq \mathrm{Int}_\gamma(A)$. Similarly, $F \wedge \mathrm{Cl}_\gamma(B) \leq \mathrm{Int}_\gamma(B)$. Now we have

$$\begin{aligned} F = F \wedge (A \vee B) &\leq (F \wedge \mathrm{Cl}_\gamma(A)) \vee (F \wedge \mathrm{Cl}_\gamma(B)) \\ &\leq \mathrm{Int}_\gamma(A) \vee \mathrm{Int}_\gamma(B) \\ &\leq \mathrm{Int}_\gamma(A \wedge B). \end{aligned}$$

Hence $F \leq \mathrm{Int}_\gamma(A \vee B)$ and thus $A \vee B$ is g*f. γ-open.
(b): The proof is similar to (a).

Remark 3.4 The union of two g*f. γ-open (resp. gf. γ-open) sets is generally not g*f. γ-open (resp. gf. γ-open) (see Example 2.5).

Theorem 3.5 Let γ be an open operation.
(a) If $\mathrm{Int}_\gamma(A) \leq B \leq A$ and if A is g*f. γ-open, then B is g*f. γ-open.
(b) If $\mathrm{Int}_\gamma(A) \leq B \leq A$ and if A is gf. γ-open, then B is gf. γ-open.

Corollary 3.6 Let γ be an open operation.
(a) If $\tau_\gamma\text{-}\mathrm{Int}_\gamma(A) \leq B \leq A$ and if A is g*f. γ-open, then B is g*f. γ-open.
(b) If $\tau_\gamma\text{-}\mathrm{Int}_\gamma(A) \leq B \leq A$ and if A is gf. γ-open, then B is gf. γ-open.

4. Preserving on gf. γ-closed and g*f. γ-closed sets

In this section we introduce the notions of fuzzy γ- $T_{1/2}$ and fuzzy γ- T_* spaces and study their basic the properties by using the concept of gf. γ-closed and g*f. γ-open set.

Definition 4.1 A fts (X, γ) is called
(a) fuzzy γ- $T_{1/2}$ if every g*f. γ-closed set is fuzzy γ-closed.
(b) fuzzy γ- T_* if every gf. γ-closed set is fuzzy γ-closed.

Theorem 4.2 For a fts (X, τ) the following are true:
(a) If (X, τ) is fuzzy γ- T_*, then $\{x_\alpha\}$ is fuzzy closed

or fuzzy γ-open in (X, τ) for each fuzzy point x_α .
(b) If (X, τ) is fuzzy γ- $T_{1/2}$, then $\{x_\alpha\}$ is fuzzy γ-closed or fuzzy γ-open in (X, τ) for each fuzzy point x_α .

Proof. (a): If $\{x_\alpha\}$ is not fuzzy closed, then 1- $\{x_\alpha\}$ is not fuzzy open and thus gf. γ-closed. By hypothesis, 1- $\{x_\alpha\}$ is fuzzy γ-closed, i.e. $\{x_\alpha\}$ is fuzzy γ-open.
(b): The proof is similar to (a).

Every fuzzy γ- $T_{1/2}$ space is fuzzy γ- T_* but the reverse implication is not true.

Example 4.3 Let X={a,b,c} and $\tau=\{1_X, 0_X, A_1, A_2, A_3, A_4\}$ where

$$A_1(a)=1,\ A_1(b)=A_1(c)=0;$$
$$A_2(a)=A_2(c)=0,\ A_2(b)=1;$$
$$A_3(a)=A_3(b)=1,\ A_3(c)=0;$$
$$A_4(a)=A_4(c)=1,\ A_4(b)=0.$$

Let $\gamma: \tau \to I^X$ be an operation defined by $0_X{}^\gamma = 0_X$, $A^\gamma = A$ if $A=A_1$ and $A^\gamma = 1_X$ if $(0_X \neq) A \neq A_1$. Then (X, τ) is fuzzy γ- T_* space but not fuzzy γ- $T_{1/2}$.

Throughout the rest of this section, let (X, τ) and (Y, σ) be fuzzy topological space and let $\gamma: \tau \to I^X$ and $\beta: \sigma \to I^Y$ be operations on τ and σ, respectively. Let id be an identity operation.

Definition 4.4 A mapping $f: (X, \tau) \to (Y, \sigma)$ said to be
(a) fuzzy (γ, β)-continuous [7] if for each fuzzy point x_α in X and each fo-q-nbd V containing $f(x_\alpha)$, there exists a fo-q-nbd U of x_α such that $f(U^\gamma) \leq V^\beta$;
(b) fuzzy (γ, β)-closed if for any fuzzy γ-closed F of (X, τ), $f(F)$ is fuzzy β-closed in (Y, σ).

Proposition 4.5 Suppose that $f: (X, \tau) \to (Y, \sigma)$ is a fuzzy (id, β)-closed mapping.
(a) If A is gf. γ-closed in X and if f is fuzzy continuous, then $f(A)$ is gf. β-closed in Y.
(b) If A is g*f. γ-closed in X and if f is fuzzy (γ, id)-continuous, then $f(A)$ is gf. β-closed in Y.
(c) If A is gf. γ-closed in X and if f is fuzzy (id, β)-continuous, then $f(A)$ is g*f. β-closed in Y.

Theorem 4.6 Suppose that $f: (X, \tau) \to (Y, \sigma)$ is fuzzy (id, β)-closed injective mapping.
(a) If (Y, σ) is fuzzy β- T_* and if f is fuzzy continuous and fuzzy (γ, β)-continuous, then (X, τ) is fuzzy γ- T_*.
(b) If (Y, σ) is fuzzy β- T_* and if f is fuzzy (γ, id)-continuous, then (X, τ) is fuzzy γ- $T_{1/2}$.
(c) If (Y, σ) is fuzzy β- $T_{1/2}$ and if f is fuzzy(γ, β)-continuous, then (X, τ) is fuzzy γ- T_*.

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