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퍼지 집합 공간 위에서의 Skorokhod metric의 성질

Some properties of the Skorokhod metric on the space of fuzzy sets

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ABSTRACT

In this paper, we investigate some properties of the Skorokhod metric on the space $F(R^p)$ of upper semicontinuous fuzzy subsets of R^p with compact support, which include the continuity of operations, the translation invariance and convexity.

Keywords : Fuzzy Numbers, The Skorokhod metric, Translation Invariance, Convex Sets.

Ⅰ. Introduction

The Skorokhod metric on the space $D[0,1]$ of functions from $[0,1]$ into the real line R which are right-continuous and have left limits was introduced to study limits theorems for stochastic processes with jumps. It turned out that the Skorokhod metric plays a key role for the convergence of probability measures on $D[0,1]$. Recently, Joo and Kim [6] introduced a new metric d_s on the space $F(R^p)$ of upper-semicontinuous fuzzy sets in R^p with compact support which is similar to the Skorokhod metric on $D[0,1]$, and proved that $F(R^p)$ is separable and topologically complete in the metric d_s.
It is expected that the metric d_s will play an important role for the convergence of fuzzy random variables. Thus it seems to be important that we study several properties of the metric d_s.

In this paper, we investigate some properties of the metric d_s on the space $F(R^p)$ which include the continuity of operations, the translation invariance and convexity.

II. Preliminaries

Let $K(R^p)$ denote the family of non-empty compact subsets of the Euclidean space R^p. Then the space $K(R^p)$ is metrizable by the Hausdorff metric h defined by

$$h(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\}.$$

A norm of $A\in K(R^p)$ is defined by

$$\|A\| = h(A,\{0\}) = \sup_{a\in A}|a|.$$

The linear structure on $K(R^p)$ is defined as usual:

$$A\oplus B = \{a+b : a\in A, b\in B\},$$
$$\lambda A = \{\lambda a : a\in A\}$$

for $A,B\in K(R^p)$ and $\lambda\in R$. In particular, if $p=1$, the multiplication on $K(R)$ can be defined as follows:

$$A\odot B = \{ab : a\in A, b\in B\}.$$

We denote by $K_c(R^p)$ the family of all convex $A\in K(R^p)$.

Now let us denote by $F(R^p)$ the family of all fuzzy sets $\tilde{u}:R^p\to[0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x\in R^p$ such that $\tilde{u}(x)=1$;

(2) $\tilde{u}$ is upper semicontinuous;

(3) supp $\tilde{u}$ = $\overline{\{x\in R^p : \tilde{u}(x)>0\}}$ is compact.

A fuzzy set $\tilde{u}\in F(R^p)$ is called a fuzzy number if it is a convex fuzzy set, i.e.,

$$\tilde{u}(\lambda x+(1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$$

for $x,y\in R^p$ and $\lambda\in[0,1]$. The family of all fuzzy numbers in R^p will de denoted by $F_c(R^p)$.

For a fuzzy set $\tilde{u}$ in R^p, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha\tilde{u} = \begin{cases}\{x : \tilde{u}(x)\geq\alpha\} & \text{if } 0<\alpha\leq 1,\\ \text{supp }\tilde{u} & \text{if } \alpha=0.\end{cases}$$

Then, it follows immediately that

$\tilde{u}\in F(R^p)$ if and only if $L_\alpha\tilde{u}\in K(R^p)$

and

$\tilde{u}\in F_c(R^p)$ if and only if $L_\alpha\tilde{u}\in K_c(R^p)$

for each $\alpha\in[0,1]$.

The linear structure on $F(R^p)$ is defined as usual;

$$(\tilde{u}\oplus\tilde{v})(z) = \sup_{x+y=z}\min(\tilde{u}(x),\tilde{v}(y)),$$

$$(\lambda\tilde{u})(z) = \begin{cases}\tilde{u}(z/\lambda) & \text{if } \lambda\neq 0,\\ I_{\{0\}} & \text{if } \lambda=0,\end{cases}$$

for $\tilde{u},\tilde{v}\in F(R^p)$ and $\lambda\in R$, where $I_{\{0\}}$ is the indicator function of $\{0\}$. Also, for the case of $p=1$, the multiplication on $F(R)$ can be defined as follows:

$$(\tilde{u}\odot\tilde{v})(z) = \sup_{xy=z}\min(\tilde{u}(x),\tilde{v}(y)),$$

Now, in order to generalize the Hausdorff metric h on $K(R^p)$ to $F(R^p)$, we define the metric d_∞ on $F(R^p)$ by

$$d_\infty(\tilde{u},\tilde{v}) = \sup_{0\leq\alpha\leq 1} h(L_\alpha\tilde{u}, L_\alpha\tilde{v})$$

Also, the norm of $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, I_{\{0\}}) = \sup_{x\in L_0\tilde{u}}|x|.$$

Then it is well-known that $F(R^p)$ is complete but non-separable w.r.t. the metric d_∞(see Klement et. al. [10]). Recently, Joo and Kim [6] introduced a new metric d_s on $F(R^p)$ which makes it a separable metric space as follows:

Definition 2.1. Let T denote the class of strictly increasing, continuous mappings of [0,1] onto itself. For $\tilde{u}, \tilde{v} \in F(R^p)$, we define

$$d_s(\tilde{u}, \tilde{v}) = \inf \{ \varepsilon > 0 : \text{there exists a } t \in T \text{ such that } \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \varepsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \varepsilon \},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

Then it follows immediately that d_s is a metric on $F(R^p)$ and $d_s(\tilde{u}, \tilde{v}) \le d_\infty(\tilde{u}, \tilde{v})$. The metric d_s will be called the Skorokhod metric. Note that a sequence $\{\tilde{u}_n\} \in F(R^p)$ converges to a limit $\tilde{u}$ in the metric d_s if and only if there exists a sequence of functions $\{t_n\}$ in T such that

$$\lim_{n \to \infty} t_n(\alpha) = \alpha \text{ uniformly in } \alpha,$$

and

$$\lim_{n \to \infty} d_\infty(t_n(\tilde{u}_n), \tilde{u}) = 0.$$

If $d_\infty(\tilde{u}_n, \tilde{u}) \to 0$, then $d_s(\tilde{u}_n, \tilde{u}) \to 0$. But, the converse is not true (for counter-example, see Joo and Kim [5]).

III. Main Results

In this section, we investigate some properties of the metric d_s on the space $F(R^p)$.

Proposition 3.1.

For $\tilde{u}, \tilde{v} \in F(R^p)$ and $\lambda \in R$, we have

$$d_s(\lambda\tilde{u}, \lambda\tilde{v}) \ge |\lambda|\, d_s(\tilde{u}, \tilde{v}) \text{ if } |\lambda| \le 1,$$

$$d_s(\lambda\tilde{u}, \lambda\tilde{v}) \le |\lambda|\, d_s(\tilde{u}, \tilde{v}) \text{ if } |\lambda| \ge 1.$$

Proposition 3.2.

(1) The scalar multiplication on $(F(R^p), d_s)$ is continuous.

(2) The addition on $(F(R^p), d_s)$ is not continuous. But if $d_s(\tilde{u}_n, \tilde{u}) \to 0$, $d_s(\tilde{v}_n, \tilde{v}) \to 0$ and $d_s(\tilde{u}_n \oplus \tilde{v}_n, \tilde{w}) \to 0$, then $\tilde{w} = \tilde{u} \oplus \tilde{v}$.

(3) The multiplication on $(F(R), d_s)$ is not continuous. But if $d_s(\tilde{u}_n, \tilde{u}) \to 0$, $d_s(\tilde{v}_n, \tilde{v}) \to 0$ and $d_s(\tilde{u}_n \odot \tilde{v}_n, \tilde{w}) \to 0$ for $\tilde{u}_n, \tilde{v}_n \in F_c(R)$, then

$$\tilde{w} = \tilde{u} \odot \tilde{v}.$$

Proposition 3.3.

(1) The metric d_s is not translation invariant.

(2) $d_s(\tilde{u} \oplus \tilde{a}, \tilde{v} \oplus \tilde{b}) \le d_s(\tilde{u}, \tilde{v}) + \|\tilde{a}\| + \|\tilde{b}\|$ for any $\tilde{u}, \tilde{v}, \tilde{a}, \tilde{b} \in F(R^p)$.

(3) $\|\tilde{u}\| \le \|\tilde{v}\| + d_s(\tilde{u}, \tilde{v})$ for $\tilde{u}, \tilde{v} \in F(R^p)$.

Proposition 3.4.

(1) $d_s(\tilde{u}_1, \tilde{u}_2) \le d_s(\tilde{v}_1, \tilde{v}_2)$ does not imply $d_\infty(\tilde{u}_1, \tilde{u}_2) \le d_\infty(\tilde{v}_1, \tilde{v}_2)$.

(2) $d_\infty(\tilde{u}_1, \tilde{u}_2) \le d_\infty(\tilde{v}_1, \tilde{v}_2)$ does not imply $d_s(\tilde{u}_1, \tilde{u}_2) \le d_s(\tilde{v}_1, \tilde{v}_2)$.

Proposition 3.5. In the metric space $(F_c(R^p), d_s)$,

(1) The closure of a convex set need not be convex.

(2) The closed convex hull of a compact set need not be compact.

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