퍼지 및 지능시스템학회 논문지 2001, Vol. 11, No. 2, pp. 115-118

$L-R$ 퍼지수의 집합-이론적 연산과 제2형 퍼지집합의 기수

The set-theoretic operations of L-R fuzzy numbers and cardinalities of type-two fuzzy sets.

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요 약

본논문에서는 $L-R$퍼지수의 집합-이론적 연산의 개념을 정의하고, 이들 개념의 성질들을 조사한다. 이들 연산들의 결과들을 이용하여 제2형 퍼지집합의 기수개념에 관하여 연구한다.

ABSTRACT

In this paper, we define concepts of some set-theoretical operations of $L-R$ fuzzy numbers and discuss some properties of these concepts. Using these results, we discuss a concept of cardinality of type-two fuzzy sets.

Key Words : L-R fuzzy number, fuzzy cardinality, type-two fuzzy sets, type-two fuzzy cardinality.

1. Introduction

Let $X=\{x_1, x_2, \cdots, x_n\}$ be a finite set. A fuzzy set A in X is defined by

$$A=\{(x, \mu_A(x)) \mid x\in X\}$$

where $\mu_A : X \to [0,1]$ is the membership function of A. The complement $\overline{A}$ of a fuzzy set A has the membership function $\mu_{\overline{A}}=1-\mu_A$. Zadeh[10], A. Ralescu[7], D. Ralescu[8], Dubois and Prade[1,2,3], and Yager[14] introduced concepts of cardinality of a fuzzy set and obtained some properties of these concepts. In this paper, we define the set-theoretic operations max*, min*, and complement of $L-R$ type fuzzy numbers on $[0,1]$ and investigate some properties of these operations. Using these properties, we define a new concept of cardinality of type-two fuzzy sets and verify that this definition is a generalized concept of the cardinality of a fuzzy set. Furthermore, we obtain some results of this concept. In Section 2, we introduce the set-theoretic operations of $L-R$ fuzzy numbers and discuss some properties of these operations. In section 3, by using these properties in section 2, we define a concept of cardinality of a type-two fuzzy set and discuss some results of type-two fuzzy cardinalities.

접수일자 : 2000년 12월 6일
완료일자 : 2001년 2월 8일

2. Preliminaries and definitions.

In this section, we introduce fuzzy numbers , $L-R$ fuzzy numbers, and some operations of $L-R$ type fuzzy numbers.

Definition 2.1 [2] A fuzzy set M of $[0,1]$ is called a fuzzy number if

(1) M is normal, i.e. $\exists_1\, x_0\in[0,1]$ such that $\mu_M(x_0)=1$;
(2) M is convex, i.e. $\mu_M(\lambda x+(1-\lambda)y)\geq \min(\mu_M(x), \mu_M(y))$ for all $x,y,\lambda\in[0,1]$;
(3) μ_M is piecewise continuous.

Let L and R be strictly decreasing continuous functions from $[0,1]$ to $[0,1]$ such that $L(0)=R(0)=1$ and $L(1)=R(1)=0$. Then, L and R is called the left and the right shape function, respectively(see [6]).

Definition 2.2 A fuzzy number M of $[0,1]$ is said to be an $L-R$ fuzzy number, $M=(m, \alpha, \beta)_{LR}$ if its membership function is defined by

(i) When $\alpha>0$ and $\beta>0$,

$$\mu_M(x)=\begin{cases} L\left(\frac{m-x}{\alpha}\right) & \text{for } m-\alpha\leq x\leq m\leq 1, x\geq 0, \\ R\left(\frac{x-m}{\beta}\right) & \text{for } m+\beta\geq x\geq m\geq 0, x\leq 1, \\ 0 & \text{else.} \end{cases}$$

(ii) When $\alpha=0$ and $\beta>0$,

$$\mu_M(x)=\begin{cases} R\left(\frac{x-m}{\beta}\right) & \text{for } m+\beta\geq x\geq m\geq 0, x\leq 1,\\ 0 & \text{else.}\end{cases}$$

(iii) When $\alpha>0$ and $\beta=0$,

$$\mu_M(x)=\begin{cases} L\left(\frac{m-x}{\alpha}\right) & \text{for } m-\alpha\leq x\leq m\leq 1, x\geq 0,\\ 0 & \text{else.}\end{cases}$$

(i) When $\alpha=0$ and $\beta=0$,

$$\mu_M(x)=\begin{cases} 1 & \text{for } x=m,\\ 0 & \text{else.}\end{cases}$$

Here, α and β are called the left and the right spreads of an $L-R$ fuzzy number M, respectively.

A_{LR} will stand for the class of all $L-R$ fuzzy numbers of of $[0,1]$.

Definition 2.3 Let $M=(m,\alpha,\beta)_{LR}$ and $N=(n,\gamma,\delta)_{LR}$ be elements fo A_{LR}. Then $\max^*(M,N)$, $\min^*(M,N)$ are defined by

$$\max{}^*\{M,N\}=(m\vee n, \alpha\wedge\beta, \beta\vee\delta)_{LR},$$

$$\min{}^*\{M,N\}=(m\wedge n, \alpha\vee\beta, \beta\wedge\delta)_{LR},$$

where $x\vee y$ and $x\wedge y$ are $\max\{x,y\}$ and $\min\{x,y\}$, respectively.

Theorem 2.4 Let $M=(m,\alpha,\beta)_{LR}$ and $N=(n,\gamma,\delta)_{LR}$ be elements of A_{LR}. Then, we have $\max^*(M,N)=M$, $\min^*(M,N)=N$ if and only if $m\leq n$, $\alpha\geq\gamma$, and $\beta\leq\delta$.

Proof.

$\max^*\{M,N\}=M$, $\min^*\{M,N\}=N$

$\Leftrightarrow (m\wedge n, \alpha\vee\gamma, \beta\wedge\delta)_{LR}$

$=(m,\alpha,\beta)_{LR}$ and $(m\vee n, \alpha\wedge\gamma, \beta\vee\delta)_{LR}=(n,\gamma,\delta)_{LR}$

$\Leftrightarrow m\wedge n=m$

and $m\vee n=n$, $\alpha\vee\gamma=\alpha$ and $\alpha\wedge\gamma=\gamma$, $\beta\wedge\delta=\beta$

and $\beta\vee\delta=\delta$

$\Leftrightarrow m\leq n$, $\alpha\geq\gamma$, and $\beta\leq\delta$.

In order to define the complement of $L-R$ fuzzy numbers, we introduce the following definition of subtraction of fuzzy numbers.

Definition 2.5 [2,3] Let M and N be fuzzy numbers on $[0,1]$. Then, the subtraction of M and N is a function $M\ominus N: [0,1]\to[0,1]$ with its membership function defined by

$$\mu_{M\ominus N}(z)=\sup_{z=x-y}\mu_M(x)\wedge\mu_N(y) \quad \text{for all } z\in[0,1]$$

By using Definition 2.5 and Eq.(4.2.1) of [3], we can obtain the following theorem.

Theorem 2.6 [3] (1) If $M=(m,\alpha,\beta)_{LR}$ and $N=(n,\gamma,\delta)_{RL}$ is a element of A_{LR} and A_{RL}, respectively, then $M\ominus N=(m-n,\alpha+\gamma,\beta+\delta)_{LR}$.

(2) If $M=(m,\alpha,\beta)_{LR}$ and $\tilde{1}=(1,0,0)_{RL}$ is a element of A_{LR} and A_{RL}, respectively, then $\tilde{1}\ominus M=(1-m,\beta,\alpha)_{RL}$.

From Theorem 2.6(2), we can the following complement of $L-R$ type fuzzy numbers.

Definition 2.7 Let $M=(m,\alpha,\beta)_{LR}$ and $\tilde{1}=(1,0,0)_{RL}$ be a element of A_{LR} and A_{RL}, respectively. The complement $\overline{M^*}$ of M is defined by $\overline{M^*}\equiv\tilde{1}\ominus M=(1-m,\beta,\alpha)_{RL}$.

3. Type-two fuzzy cardinalities.

Let $X=\{x_1, x_2,\cdots,x_n\}$ be a finite set. Using the set-theoretic operations of $L-L$ type fuzzy numbers on $[0,1]$, we define the following new concept of type-two fuzzy cardinality.

Definition 3.1 Let $F: X\to A_{LL}$ be a type-two fuzzy set and $F(x_k)=M_k=(m_k,\alpha_k,\beta_k)_{LL}$ for $k=1,2,\cdots,n$. Then, a type-two fuzzy cardinality of F is a function f_2-card $F: \{0,1,\cdots,n\}\to A_{LL}$ defined by

$$f_2\text{-card } F(k)=\min{}^*\{M_{(k)}, \overline{M^*}_{(k+1)}\} \text{ for } k=1,2,\cdots,n$$

where $M_{(1)},M_{(2)},\cdots,M_{(n)}$ are $L-L$ type fuzzy numbers of $M_1,M_2,\cdots,M_n$ arranged in decreasing order of magnitude of the normal points m_k, for $k=1,2,\cdots,n$, and $M_{(0)}=(1,0,0)_{LL}$, $M_{(n+1)}=(0,1,0)_{LL}$.

From Definition 3.1, we can obtain the following proposition.

Proposition 3.2 Let $F: X\to A_{LL}$ be as in Definition 3.1. Then we have that f_2-card $F(k)=(m_{(k)}\wedge(1-m_{(k+1)}), \alpha_{(k)}\vee\beta_{(k+1)}, \beta_{(k)}\wedge\alpha_{(k+1)})_{LL}$ for $k=0,1,\cdots,n$, where $m_{(0)}=1$, $m_{(n+1)}=0$.

Proof. Since $M_{(k+1)}\in A_{LL}$, by using Theorem 2.6(2), we have

$$\overline{M^*}_{(k+1)}=(1-m_{(k+1)},\beta_{(k+1)},\alpha_{(k+1)})_{LL}.$$

Then, we have

$$\begin{aligned} f_2\text{-card}F(k) &= \min{}^*\{M_{(k)}, \overline{M^*}_{(k+1)}\}\\ &=(m_{(k)}\wedge(1-m_{(k+1)}), \alpha_{(k)}\vee\beta_{(k+1)},\\ &\quad \beta_{(k)}\wedge\alpha_{(k+1)})_{LL}.\end{aligned}$$

Remark 3.3 If

$$F(k)=\begin{cases}(1,0,0)_{LL} & \text{if } k=1,2,\cdots,r\\(0,0,0)_{LL} & \text{if } k=r+1,\cdots,n\end{cases},$$

then F is a nonfuzzy set with r elements by convention.

Theorem 3.4 Let $F: X\to A_{LL}$ be a type-two fuzzy set. Then,

$$f_2-\text{card}F(k)=\begin{cases}(1,0,0)_{LL} & \text{if } k=r\\(0,0,0)_{LL} & \text{if } k\neq r\end{cases}$$

if and only if F is a nonfuzzy set with r elements.

Proof. Suppose that

$$f_2-\text{card}F(k)=\begin{cases}(1,0,0)_{LL} & \text{if } k=r\\(0,0,0)_{LL} & \text{if } k\neq r\end{cases}$$

When $k=r$, we have

$$(m_{(r)}\wedge(1-m_{(r+1)}),\alpha_{(r)}\vee\beta_{(r+1)},\beta_{(r)}\wedge\alpha_{(r+1)})_{LL}$$
$$=(1,0,0)_{LL}.$$

Thus, we have

$$m_{(r)}\wedge(1-m_{(r+1)})=1,\ \alpha_{(r)}\vee\beta_{(r+1)}=0,$$
$$\text{and } \beta_{(r)}\wedge\alpha_{(r+1)}=0.$$

Since $m_{(0)}\geq m_{(1)}\geq\cdots\geq m_{(n+1)}$ and $\alpha_{(k)}\vee\beta_{(k+1)}=0$, $m_{(0)}=m_{(1)}=\cdots=m_{(r)}=1$ and $m_{(r+1)}=\cdots=m_{(n+1)}=0$, and

$$\alpha_{(r)}=\beta_{(r+1)}=0 \qquad (3\text{-}1)$$

When $k\neq r$, we have

$$(m_{(k)}\wedge(1-m_{(k+1)}),\alpha_{(k)}\vee\beta_{(k+1)},\beta_{(k)}\wedge\alpha_{(k+1)})_{LL}$$
$$=(0,0,0)_{LL}.$$

Then, $\alpha_{(k)}\vee\beta_{(k+1)}=0$ and hence

$$\alpha_{(k)}=\beta_{(k+1)}=0 \qquad \text{for all } k\neq 0 \qquad (3\text{-}2)$$

From (3 1) and (3-2), $\alpha_{(1)}=\alpha_{(2)}=\cdots=\alpha_{(n)}=0$ and $\beta_{(1)}=\beta_{(2)}=\cdots=\beta_{(n)}=0$. If we put $F(k)=M_k$, then we have

$$M_{(k)}=\begin{cases}(1,0,0)_{LL} & \text{if } k=1,2,\cdots,r\\(0,0,0)_{LL} & \text{if } k=r+1,\cdots,n\end{cases},$$

By Remark 3.3, F is a nonfuzzy set with r elements.

Conversely, if we put $F(x_k)=M_k=(m_k,0,0)_{LL}$ for $k=1,2,\cdots,n$, by Remark 3.3 and the hypothesis,

$$M_{(k)}=(m_{(k)},\alpha_{(k)},\beta_{(k)})_{LL}=\begin{cases}(1,0,0)_{LL} & \text{if } k=1,2,\cdots,r\\(0,0,0)_{LL} & \text{if } k=r+1,\cdots,n\end{cases},$$

Thus, $m_{(0)}=m_{(1)}=\cdots=m_{(r)}=1$, $m_{(r+1)}=\cdots=m_{(n+1)}=0$, and $\alpha_{(k)}=\beta_{(k)}=0$ for $k=1,2,\cdots,n$. If we set $m_{(0)}=1$, $m_{(n+1)}=0$, $\alpha_{(0)}=\alpha_{(n+1)}=\beta_{(0)}=\beta_{(n+1)}=0$, by using the definition of f_2-card $F(k)$, we have

$$f_2-\text{card}F(k)=\begin{cases}(1,0,0)_{LL} & \text{if } k=r\\(0,0,0)_{LL} & \text{if } k\neq r\end{cases}$$

Lemma 3.5 Let $\{\alpha_{(k)}\}_{k=1}^{n}$ be a sequence in $[0,1]$ and $\alpha(1)\geq\alpha_{(2)}\geq\cdots\geq\alpha_{(n)}\geq 0$. If $k\geq l$, then we have $\alpha_{(k)}\geq\alpha_{(k+1)}\geq\alpha_{(l)}$.

Proof. If $k\geq l$, we have either $k+1=l$ or $k+1<l$. When $k+1=l, \alpha_{(k+1)}=\alpha_{(k)}$; when $k+1<l, \alpha_{(k+1)}\geq\alpha_{(l)}$. Thus, $\alpha_{(k+1)}\geq\alpha_{(l)}$ and hence $\alpha_{(k)}\geq\alpha_{(k+1)}\geq\alpha_{(l)}$.

Definition 3.6 Let $M=(m,\alpha,\beta)_{LL}$ and $N=(n,\gamma,\delta)_{LL}$ be elements of A_{LL}. Then, we define the order $\leq$ of M and N ;

$$M\leq N \text{ if and only if } m\leq n,\ \alpha\geq\gamma,\ \text{and } \beta\leq\delta.$$

We remark that the condition $M_{(1)}\geq M_{(2)}\geq\cdots\geq M_{(n)}$ is equivalent to the following condition

$$m_{(1)}\geq m_{(2)}\geq\cdots\geq m_{(n)},\ \alpha_{(1)}\leq\alpha_{(2)}\leq\cdots\leq\alpha_{(n)},$$
$$\text{and } \beta_{(1)}\geq\beta_{(2)}\geq\cdots\geq\beta_{(n)}.$$

Definition 3.7 Let $G: \{0,1,\cdots,n\}\to A_{LL}$ be a type-two fuzzy set. Then, G is said to be convex if

$$G(l)\geq\min{}^*\{G(k),G(r)\} \text{ whenever } k\leq l\leq r.$$

Theorem 3.8 Let $F: X\to A_{LL}$ be a $L-L$ type fuzzy number and $F(x_k)=M_k=(m_k,\alpha_k,\beta_k)_{LL}$ for $k=1,2,\cdots,n$. Assume that $M_{(1)}\geq M_{(2)}\geq\cdots\geq M_{(n)}$. Then, f_2-card F is convex.

Proof. When $k\leq l\leq r$, we should prove that

$$f_2-\text{card}F(l)\geq\min{}^*\{f_2-\text{card}F(k),f_2-\text{card}F(r)\}.$$

By Definition 3.7, it is sufficient to show that

(i) $\{m_{(k)}\wedge(1-m_{(k+1)})\}\wedge\{m_{(r)}\wedge(1-m_{(r+1)})\}\leq m_{(l)}\wedge(1-m_{(l+1)})$,

(ii) $\{\alpha_{(k)}\vee\beta_{(k+1)}\}\vee\{\alpha_{(r)}\vee\beta_{(r+1)}\}\geq\alpha_{(l)}\vee\beta_{(l+1)}$, and

(iii) $\{\beta_{(k)}\wedge\alpha_{(k+1)}\}\wedge\{\beta_{(r)}\wedge\alpha_{(r+1)}\}\leq\beta_{(l)}\wedge\alpha_{(l+1)}$.

Since $m_{(k)}\geq m_{(l)}\geq m_{(r)}$ when $k\leq l\leq r$, by using Lemma 3.5,

$$m_{(k)}\geq m_{(k+1)}\geq m_{(l)}\geq m_{(l+1)}\geq m_{(r)}\geq m_{(r+1)}.$$

Thus, we have

$$\{m_{(k)}\wedge(1-m_{(k+1)})\}\wedge\{m_{(r)}\wedge(1-m_{(r+1)})\}$$
$$\leq(1-m_{(k+1)})\wedge m_{(r)}$$
$$\leq(1-m_{(l+1)})\wedge m_{(l)}$$
$$=m_{(l)}\wedge(1-m_{(l+1)})$$

Hence (i) holds. Since $\alpha_{(k)} \le \alpha_{(l)} \le \alpha_{(r)}$ and $\beta_{(k)} \ge \beta_{(l)} \ge \beta_{(r)}$ when $k \le l \le r$, by using Lemma 3.5, we have

$$\alpha_{(k+1)} \le \alpha_{(k)} \le \alpha_{(l+1)} \le \alpha_{(l)} \le \alpha_{(r+1)} \le \alpha_{(r)}$$

and

$$\beta_{(k+1)} \ge \beta_{(k)} \ge \beta_{(l+1)} \ge \beta_{(l)} \ge \beta_{(r+1)} \ge \beta_{(r)}.$$

Then, we have

$$\{\alpha_{(k)} \vee \beta_{(k+1)}\} \vee \{\alpha_{(r)} \vee \beta_{(r+1)}\} \ge \alpha_{(k)} \vee \beta_{(r+1)} \ge \alpha_{(l)} \vee \beta_{(l+1)},$$

and

$$\beta_{(k)} \wedge \alpha_{(k+1)}\} \wedge \{\beta_{(r)} \wedge \alpha_{(r+1)}\} \le \beta_{(k)} \wedge \alpha_{(r+1)} \le \beta_{(l)} \wedge \alpha_{(l+1)}.$$

Hence (ii) and (iii) hold.

Theorem 3.9 Let $F: X \to A_{LL}$ be a type-two fuzzy set and

$$F(x_k) = M_k = (m_k, \alpha_k, \beta_k)_{LL} \text{ for } k = 1, 2, \cdots, n.$$

If $M_{(1)} \ge M_{(2)} \ge \cdots \ge M_{(n)}$, then we have

$$f_2\text{-card } \overline{F}(k) = f_2 - \text{card } F(n-k) \text{ for } k = 0, 1, \cdots, n.$$

where

$$\overline{F}^*(k) = \overline{M_k}^* = (1 - m_k, \beta_k, \alpha_k)_{LL} \text{ for } k = 1, 2, \cdots, n.$$

Proof. If we put $\overline{F}^*(k) = M'_k$ and $M'_k = (m'_k, \alpha'_k, \beta'_k)_{LL}$ for $k = 1, 2, \cdots, n$, by using the hypothesis $M_{(1)} \ge M_{(2)} \ge \cdots \ge M_{(n)}$, then we have

$$M'_{(1)} \le M'_{(2)} \le \cdots \le M'_{(n)}.$$

That is,

$$\text{(i) } m'_{(k)} = 1 - m_{(n-k+1)},$$

$$\text{(ii) } \alpha'_{(k)} = \beta_{(n-k+1)},$$

$$\text{(iii) } \beta'_{(k)} = \alpha_{(n-k+1)}.$$

From (i),(ii),and (iii), we have

$$f_2\text{-card } \overline{F}^*(k) = \min{}^*\{M'_{(k)}, \overline{M}^*_{(k+1)}\}$$
$$= (m'_{(k)} \wedge (1 - m'_{(k+1)}), \alpha'_{(k)} \vee \beta'_{(k+1)}, \beta'_{(k)} \wedge \alpha'_{(k+1)})_{LL}$$
$$= ((1 - m_{(n-k+1)}) \wedge m_{(n-k)}), \beta_{(n-k+1)} \vee \alpha_{(n-k)}, \alpha_{(n-k+1)} \wedge \beta_{(n-k)})_{LL}$$
$$= (m_{(n-k)} \wedge (1 - m_{(n-k+1)})), \alpha_{(n-k)} \vee \beta_{(n-k+1)}, \beta_{(n-k)} \wedge \alpha_{(n-k+1)})_{LL}$$
$$= f_2\text{-card} F(n-k)$$

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