Proceedings of the Autumn Conference in 2001
and the 30th Anniversary of the Korean Statistical Society

On the Strong Law of Large Numbers for Convex Tight Fuzzy Random Variables

Sang Yeol Joo[1] and Seung Soo Lee[1]

Abstract

We can obtain SLLN's for fuzzy random variables with respect to the new metric d_s on the space $F(R)$ of fuzzy numbers in R. In this paper, we obtain a SLLN for convex tight random elements taking values in $F(R)$.

Keywords: fuzzy random variables, strong law of large numbers, convex tightness.

1. INTRODUCTION

The concept of a fuzzy random variable was introduced as a generalization of random sets in to represent relationships between the outcomes of a random experiment and inexact data. Limit theorems for sums of independent fuzzy random variables have received much attentions because of its usefulness in several applied fields. This paper concerns with the strong law of large numbers which is one of limit theorems.

Strong laws of large numbers for sums of independent fuzzy random variables have been studied by several people, Kruse (1982), Miyakoski and Shimbo(1984), Klement et.al.[11], Inoue [5], Hong and Kim (1994), Molchanov[12], Kim [9]. Recently Joo and Kim [7] generalized Kolmogorov's SLLN to the case of fuzzy random variables. Furthermore, Joo and Kim [6] introduced a new metric d_s on the space $F(R)$ of fuzzy numbers in R so that d_s is separable and topologically complete, and Ghil et.al.[3] characterized compact subsets of $F(R)$. Also, Kim [10] proved that a fuzzy mapping is measurable iff it is measurable when considered as a function into the metric space $(F(R), d_s)$. Thus it is natural that we ask whether SLLN for fuzzy random variables can also be obtained with respect to the metric d_s. Thanks to these results, Joo et. al.[8] could obtain a SLLN for stationary fuzzy random variables.

In this paper, motivated by Joo and Kim [7], we establish a SLLN for convex tight fuzzy random variables using the above results.

[1]Department of Statistics, Kangwon National University, Kangwon , 200-701, Korea

2. PRELIMINARIES

Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties ;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semi-continuous.

(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

We denote the family of all fuzzy numbers by $F(R)$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \alpha = 0. \end{cases}$$

Then it follows that $\tilde{u}$ is fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u^1_\alpha, u^2_\alpha]$.

Theorem 1.1 of Goetschel and Voxman [4]implies that we can identify a fuzzy number $\tilde{u}$ with the parametrized representation $\{(u^1_\alpha, u^2_\alpha) | 0 \leq \alpha \leq 1\}$.

Now, we define the metric d_∞ on $F(R)$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}), \tag{2.1}$$

where h is the Hausdorff metric defined as

$$h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) = \max(|u^1_\alpha - v^1_\alpha|, |u^2_\alpha - v^2_\alpha|).$$

Also, the norm $||\tilde{u}||$ of fuzzy number $\tilde{u}$ will be defined as

$$||\tilde{u}|| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u^1_0|, |u^2_0|).$$

Then it is well-known that $F(R)$ is complete but nonseparable with respect to the metric d_∞ . Joo and Kim [6] introduced a metric d_s in $F(R)$ which makes it a separable metric space as follows ;

Definition 2.1. *Let T denote the class of strictly increasing, continuous mapping of* [0,1] *onto itself. For $\tilde{u}, \tilde{v} \in F(R)$, we define*

$$\begin{aligned} d_s(\tilde{u}, \tilde{v}) &= \inf\{\varepsilon : \textit{there exists a } t \textit{ in } T \textit{ such that} \\ &\quad \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \varepsilon \textit{ and } d_\infty(\tilde{u}, t \circ \tilde{v}) \le \varepsilon\}, \end{aligned} \tag{2.2}$$

where $t \circ \tilde{v}$ denotes the composition of $\tilde{v}$ and t.

Then it follows immediately that d_s is a metric on $F(R)$ and $d_s(\tilde{u}, \tilde{v}) \le d_\infty(\tilde{u}, \tilde{v})$. The metric d_s will be called the Skorokhod metric.

Theorem 3.2 and 3.4 in Ghil, Joo and Kim [3] which characterize compact subsets of $F(R)$ are useful in proving the main result.

3. MAIN RESULT

In this section, we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated. Also, we denote by $\mathcal{B}_s$ the Borel σ-field of $F(R)$ generated by the metric d_s.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable if it is measurable, i.e.,

$$\tilde{X}^{-1}(B) = \{\omega : \tilde{X}(\omega) \in B\} \in \mathcal{A} \text{ for every } B \in \mathcal{B}_s.$$

If we denote $\tilde{X}(\omega) = \{(X_\alpha^1(\omega), X_\alpha^2(\omega)) | 0 \le \alpha \le 1\}$, then it is known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are random variables in the usual sense (See Kim [15]). A fuzzy random variable $\tilde{X} = \{(X_\alpha^1, X_\alpha^2) | 0 \le \alpha \le 1\}$ is called integrable if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are integrable, equivalently, $\int ||\tilde{X}|| dP < \infty$. In this case, the expectation of $\tilde{X}$ is the fuzzy number $E\tilde{X}$ defined by

$$E\tilde{X} = \{(EX_\alpha^1, EX_\alpha^2) \mid 0 \le \alpha \le 1\}. \tag{3.1}$$

Definition 3.1. *A sequence $\{\tilde{X}_n\}$ of fuzzy random variables is said to be convex tight if for each $\epsilon > 0$ there is a convex compact subset K of $F(R)$ such that*

$$P(\tilde{X}_n \notin K) < \epsilon \text{ for all } n.$$

Now we propose our main theorem.

Theorem 3.1. *Let $\{\tilde{X}_n\}$ be a sequence of independent and convex tight fuzzy random variables. If*

$$\sup_n E||\tilde{X}_n||^p \le M < \infty \text{ for some } p > 1, \tag{3.2}$$

then

$$\lim_{n\to\infty} d_s\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, \frac{1}{n}\sum_{i=1}^{n}E\tilde{X}_i\right) = 0 \text{ a.s.}$$

Remark 1. In the above theorem, we assume that $\{\tilde{X}_n\}$ is convex. The need of convexity arises from the desired condition that a convex combination of elements $\{\tilde{u}\}$ of K, in particular, $\frac{1}{n}\sum_{i=1}^{n}\tilde{u}_i$, again belong to K. It remains an open problem whether the similar result holds if we replace convex tightness by tightness.

REFERENCES

[1] P. Billingsley (1968), Convergence of probability measure, Second edition, Wiley.

[2] P. Z. Daffer and R. L. Taylor (1979), Laws of large numbers for $D[0,1]$, Ann. Probab. 7, 85-95.

[3] B. M. Ghil, S. Y. Joo and Y. K. Kim (2001), A characterization of compact subsets of fuzzy number space, Fuzzy Sets and Systems, 123, 191-195.

[4] R. Goetschel and W. Voxman (1986), Elementary fuzzy calculus, Fuzzy Sets and Systems, 18, 31-43.

[5] H. Inoue (1991), A strong law of large numbers for fuzzy random sets, Fuzzy Sets and Systems, 41, 285-291.

[6] S. Y. Joo and Y. K. Kim (2000), The Skorokhod tpoplogy on space of fuzzy numbers, Fuzzy Sets and Systems, 111, 497-501.

[7] S. Y. Joo and Y. K. Kim (2001), Kolmogorov's strong law of large numbers for fuzzy random variables, Fuzzy Sets and Systems, 120, 499-503.

[8] S. Y. Joo, S. S. Lee and Y. H. Yoo (2001), A strong law of large numbers for stationary fuzzy random variables, Journal of the Korean Statistical Society, 30, 153-161.

[9] Y. K. Kim (2000), A strong law of large numbers for fuzzy random variables, Fuzzy Sets and Systems, 111, 319-323.

[10] Y. K. Kim, Measurability of fuzzy valued functions, Fuzzy Sets and Systems, to appear.

[11] E. P. Klement, M. L. Puri and D. A. Ralescu (1986), Limit theorems for fuzzy random variables, Proc. Roy. Soc. London ser. A 407, 171-182.

[12] I. S. Molchanov (1999), On strong law of large numbers for random upper semicontinuous functions, J. Math. Anal. Appl. 235, 349-355.