Link Importance in Flow Network

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Abstract

The flow network is considered to be in a functioning state if it can transmit a maximum flow which is greater than or equal to a specified amount of flow. In this paper we consider the measures of importance of a link in the flow network. We define the structural importance and reliability importance of link when the required amount of flow is given. Also, we present the performance importance of link in a flow network. The performance importance can be used to determine which links should be improved first in order to make the greatest improvement in performance of the network. Numerical examples are presented for illustrative purpose.

Keywords. Flow network, Capacity, Maximum flow, Performance index, Link importance

1. Introduction

The link importance in the network attracts a great deal of interests among engineers and reliability analysts. The reliability importance of a link(edge, component) is the rate at which the network reliability changes with respect to changes in the reliability of that link. The link importance can be used to determine which links should be improved first in order to make the greatest improvement in the network reliability.

The network is represented by a probabilistic graph $G(V, E)$, which consists of a set V of nodes and a set E of links. The flow network is considered to be in a functioning state if it can transmit a maximum flow which is greater than or equal to a specified amount of flow. Each link may have different flow capacity and the network may be required to transmit a specified amount of flow from the source node to the terminal node. Examples of such networks include a computer communication network which allows only a fixed amount of data exchange among different terminals of various computer centers and a transport system of a large town which limits maximum traffic on various roads. Lee and Park(2000) present an efficient method of constructing k-minimal path sets to evaluate the reliability of a flow network.

There are several measures of importance of a link in the network. Birnbaum(1969) defines the reliability importance of component i and other authors have proposed extensions of this concept. Armstrong(1997) propose the importance measures to cover reliability models in which the components have two failure-modes rather than the conventional one failure-mode. Page and Perry(1994) examine the use of reliability polynomials to rank the edges in a graph in terms of overall importance to the graph reliability. Hong and Lie(1993) introduce the joint reliability importance(JRI) of two edges in an undirected network.

In this paper we consider the measures of importance of a link in the flow network. The structural importance and reliability importance of link in a flow network is presented when the required amount of flow is given. Also, we present the performance importance of link in a flow network. These measures are based on both reliability and capacity of the link. The performance importance can be used to determine which links should be improved first in order to make the greatest improvement in performance of the network.

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Section 2 describes the network reliability when the specified amount of flow is given. Also, the performance index of flow network is considered. Section 3 defines the structural importance, reliability importance and performance importance of link in a flow network. In Section 4, the numerical examples for the flow network are given for illustrative purpose.

Notation

ϕ	underlying coherent binary structure of network
c_i	flow capacity of link i
$\hat{c}$	capacity vector : $\hat{c} = (c_1, \cdots, c_n)$
p_i	probability that link i is functioning
$\hat{p}$	link reliability vector : $\hat{p} = (p_1, \cdots, p_n)$
X_i	random variable indicating the state of link i
$\hat{X}$	random state vector : $\hat{X} = (X_1, \cdots, X_n)$
$\hat{x}$	binary vector values that $\hat{X}$ can assume
k	required amount of flow
$S^k(\hat{x})$	structure function of flow network in level k
$R^k(\hat{p})$	probability that network is working in k-level
$PI(\hat{p})$	performance index of flow network
SI_i^k	k-level structural importance for link i
RI_i^k	k-level reliability importance for link i
PI_i	performance importance for link i

2. Network Reliability and Performance Index

Assumptions

1. The nodes are perfect and each has no capacity limit.
2. The links are independent and either function or fail with known probability.
3. All the links are directed and each link flow is bounded by the capacity of the link.
4. No information or flow can be transmitted through a failed link.
5. The network is good if and only if a required amount of flow can be transmitted from the source node to the terminal node.

Flow network is considered to be in a functioning state if it can transmit a maximum flow which is greater than or equal to a specified amount of flow, k say. Each link may have different flow capacity and the network may be required to transmit a specified amount of flow from the source node to the terminal node.

Definition 2.1. A vector $\hat{x}$ is said to be a k-minimal path vector if $M_\phi(\hat{x} : \hat{c}) \geq k$ and $M_\phi(\hat{y} : \hat{c}) < k$ for all $\hat{y} < \hat{x}$. For a k-minimal path vector $\hat{x}$, the set $A = \{i | x_i = 1\}$ is called a k-minimal path set(k-mps). The k-mps is also called a mps of level k.

In Definition 2.1, $M_\phi(\hat{x} : \hat{c})$ denotes the maximum amount of flow that can be transmitted through the network successfully, given ϕ, $\hat{x}$, and $\hat{c}$.

The probability that a network is working in k-level is defined by

$$R^k(\hat{p}) = E[S^k(\hat{x})]$$
$$= E\left[1 - \prod_{j=1}^{n} \left(1 - \rho_j^k(\hat{x})\right)\right],$$

where n is the number of k-mps and ρ_j^k is the series structure of jth k-mps.
The network reliability is the probability that the required amount of flow can be transmitted from the input node to the output node.

Now, we define the performance index of a flow network using the network reliability in k-level. The performance index for a flow network is defined as a composite index integrating the importance measures of both reliability and capacity. The performance index for a flow network is defined as

$$PI(\hat{p}) = \sum_{k=1}^{max} k[R^k(\hat{p}) - R^{k+1}(\hat{p})],$$

where $R^{max+1} = 0$. Thus, $PI(\hat{p})$ can be interpreted as the expected amount of flow that the network can transmit from the input node to the output node when each link functions with a probability p.

3. Importance Measures of Link

In this section we consider the structural importance and reliability importance of a link when the required amount of flow is given. Also, we present the performance importance of link in a flow network.

We first consider the structural importance of a link. The structural importance of a link is one of the simplest measures of link importance. The structural importance of a link i in k-level is the proportion of 2^{n-1} states of the flow network in which the functioning of link i makes the flow network function, while the failure of link i cause the failure of the network. That is, the structural importance in k-level is defined by

$$SI_i^k = 2^{-(n-1)} \cdot \sum [S^k(1_i, \hat{x}) - S^k(0_i, \hat{x})], \tag{1}$$

where $S^k(\hat{x})$ is the structural function of the network in k-level, given $\hat{x}$. Here, $(1_i, \hat{x})$ and $(0_i, \hat{x})$ denote the state vectors in which the ith link assumes 1 and 0, respectively.

Next, we consider the reliability importance of link i in k-level of the flow network. The k-level reliability importance can be used to determine which links should be improved first in order to make the greatest improvement in the flow network. That is, the k-level reliability importance is a measure of the importance of a link in determining the k-level network reliability. The k-level reliability importance considers not only the network topology, but also the link capacity and link reliability.

The reliability importance of link i in k-level is defined as follows.

$$RI_i^k = \partial R^k(\hat{p})/p_i, \tag{2}$$

where $R^k(\hat{p})$ is the probability that the network is working in k-level and p_i is the probability that link i is functioning.

If the links are independent with $p_i = 0.5$, then it turns out that $SI_i^k = RI_i^k$.

Finally, we define the performance importance of link i in a flow network. The performance importance can be used to determine which links should be improved first in order to make the greatest improvement in performance of the network.

The performance importance of link i in a flow network is defined as follows.

$$PI_i = \partial PI(\hat{p})/p_i, \tag{3}$$

where $PI(\hat{p})$ is the performance index of a flow network, which is defined in Section 2.

It is also possible that the performance importance of link i can be defined by using the k-level reliability importance as follows.

$$PI_i = \sum_{k=1}^{max} RI_i^k. \tag{4}$$

We can easily show that the definitions (3) and (4) are equivalent.

4. Numerical Examples

Example 4.1. We consider the simple network, given in Figure 4.1, with the flow capacities $c_1 = 3$, $c_2 = 2$ and $c_3 = 1$. For this particular case, the network reliability in the level $k(= 1,2,3)$ can be obtained as follows : $R^1(\hat{p}) = p_1p_2 + p_1p_3 - p_1p_2p_3$, $R^2(\hat{p}) = p_1p_2$ and $R^3(\hat{p}) = p_1p_2p_3$. If $p_1 = p_2 = p_3 = p = 0.9$, then $R^1(\hat{p}) = 0.891, R^2(\hat{p}) = 0.81, R^3(\hat{p}) = 0.729$. Also, the performance index for the simple network, $PI(\hat{p})$, is 2.43 when $p_1 = p_2 = p_3 = p = 0.9$.

Figure 4.1. Simple Network

Table 4.1 gives the values of RI_i^k and PI_i when $c_1 = 3$, $c_2 = 2$ and $c_3 = 1$. For $k = 1$, Table 4.1 shows that the reliability importance of link 1 is the largest and thus, the reliability of link 1 should be improved first in order to make the greatest improvement in reliability of the network. Also, Table 4.1 shows that reliability of link 1 should be improved first in order to make the greatest improvement in performance of the network, since the performance importance of link 1 is the largest among those of three links.

Table 4.1. Link Importance of the Simple Network

Link Number	Reliability Importance RI_i^k			Performance Importance
i	$k=1$	$k=2$	$k=3$	PI_i
1	$2p-p^2$	p	p^2	$3p$
2	$p-p^2$	p	p^2	$2p$
3	$p-p^2$	0	p^2	p

Example 4.2. We consider the bridge network of Figure 4.2. The flow capacities and link reliabilities are fixed at $c_1 = 6$, $c_2 = 2$, $c_3 = 2$, $c_4 = 3$, $c_5 = 2$ and $p_1 = p_2 = p_3 = p_4 = p_5 = p$.

Figure 4.2. Bridge Network

Table 4.2 shows the values of RI_i^k and PI_i for fixed c_i for $i = 1, \cdots, 5$. For $k = 1$, Table 4.2 shows that the reliabilities of link 1 or link 5 should be improved first in order to make the greatest improvement in reliability of the network. Note that the reliability importance of link 1 or 5 is the greatest. From Table 4.2, we also observe that $PI_1 > PI_4 > PI_5 > PI_2 > PI_3$. This implies that the reliability of link 1 should be improved first in order to improve the performance of the network the most.

Table 4.2. Link Importance of the Bridge Network

	Reliability Importance RI_i^k					Performance
i	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	Importance PI_i
1	$p+p^2-3p^3+p^4$	$p+p^2-3p^3+p^4$	p	$2p^3-p^4$	$2p^3-p^4$	$3p+2p^2-2p^3$
2	$p-2p^3+p^4$	$p-2p^3+p^4$	0	p^3-p^4	p^3-p^4	$2p-2p^3$
3	$p^2-2p^3+p^4$	$p^2-2p^3+p^4$	0	p^3-p^4	p^3-p^4	$2p^2-2p^3$
4	$p-2p^3+p^4$	$p-2p^3+p^4$	p	$2p^3-p^4$	$2p^3-p^4$	$3p$
5	$p+p^2-3p^3+p^4$	$p+p^2-3p^3+p^4$	0	$2p^3-p^4$	$2p^3-p^4$	$2p+2p^2-2p^3$

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