Evaluation of Procss Capability measres for Exponential Distributed Data

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Abstract. The main objective of this paper to purpose a evaluating methods of process capability measures for exponential distributed quality characteristics.

For correctly evaluating process capability , the first thing , exponential data is applied the Lilliefors test statistic to the null hypothesis of normality. The next , exponential parameters is estimated in terms of MLE , ME , MME and then evaluated , respectively , process capability index based on exponential curved (I_e) proposed by in this study and process capability indices based on Pearson system and Johnson system.

Key Words : process capability, process capability index based on exponential curves(I_e) , Pearson system , Johnson system.

I . Introduction

A common , assumption in capability studies in that the individuls in the process being follow a normal distribution.

If this is not the case, especially when the underlying probability distribution in heavily skewed , then the conclusions of the study are likely to be invalid.

Yet , acceptable replacements for the process capability indices based on a normal distribution are available , if we could only determine the underlying distribution.

To solve this problem , the first thing , exponential data is applied the Lilliefors test at statistic to the null hypothesis of normality.

The next exponential parameters is estimated in terms of the maximum likelihood estimators(MLE), a modification of the moment estimators(MME) and the moment estimators(ME) and then evaluated , respectively , process capabiliy index based on exponential curves(I_e) proposed by in this study and process capability indices on Pearson system and Johnson system.

Ⅱ. The Lilliefors Test for the Exponential Distribution and parameter Estimatian

The two-parameter exponential distribution has (cdf , pdf) and hf

$$F(x;\theta,\gamma)=1-\exp(-\frac{x-r}{\theta}) \quad , \quad x>r \tag{2.1}$$

$$f(x;\theta,r)=\frac{1}{\theta}\exp(-\frac{x-r}{\theta}) \tag{2.2}$$

$$h(x;\theta,r)=\frac{1}{\theta} \tag{2.3}$$

Where $\theta>0$ in a scale parameter and r is both a location and a threshold parameter. For $\gamma=0$ this in the well-known one-parameter exponential distribution.

The mean and variance of the exponential distribution are , respectively,

$$E(X)=r+\theta \tag{2.4}$$

$$Var(X)=\theta^2 \tag{2.5}$$

The P quantile of the exponential distribution is

$$x_p=r-\log(1-p)\theta \tag{2.6}$$

The exponential distribution in widely used in the field of reliability engineering as a model of the time of a component or system. In these application , the exponential distribution is a popular distribution for some kinds of electronic components as an example , capacitors or robust , high-quality integrated circuits.

But , this exponential distribution would not be appropriate for a population of electronic component having-causing quality defects.

The exponential distribution in usually inappropriate for modeling the life of mechanical components like bearing , subject to some combination of fatigue , corrosion , or wear. It is also usually inappropriate for electronic components that exhibit wearout properties over their technological life like lasers and filament devices[1].

2.1 The Lilliefors Test for the Exponential distribution

The data consist of a randome sample $X_1, X_2,..., X_n$ of size n associated with some unknown distribution function , denoted by F(x). Compute the sample mean for use as an estimate of the unknown pasameter. For each X_i , compute Z_i , defined by

$$Z_i=X_i/\overline{X} \tag{2.7}$$

for use in conputing the test statistic.

First , the empirical distribution function $S(x)$ based on $Z_1,..., Z_n$ in plotted on a graph . On the same graph the function $F^*(x)=1-e^{-x}$ is plotted for x>0 ; actually , only values at n point need to be determined , the points being at $x=Z_1$, $x=Z_2$, and so on. The maximum vertical distance between the two functions

$$T=\underset{x}{SUP} \mid F^*(x)-S(X) \tag{2.8}$$

in the test statistic [2].

2.2 parameter Estimation

(1) Maximum Likelihood Estimator (MLE)

The density approximation to the likelihood generally provides an adequate approximation for the exponential distribution. For a sample consisting of only right-censored observation and observations reported as exact failure times , it is easy to show the MLE of θ is computed as

$$\hat{\theta}=\frac{TTT}{r} \tag{2.9}$$

Where $TTT=\sum_{i=1}^{\infty}$ in known as the total time on test , and t_i , $i=1, \ldots, n$, and the reported failure times for units that failed and the running (or censoring) time for the right-censored observations[1].

(2) Moment Estimators (ME)

Moment Estimators (ME) of the exponential distribution are derived by equating sample moments to the corresponding distribution moment.

The first two moments are given by E_q.(2.4) , E_q.(2.5) and the resulting ME equations are

$$\hat{\theta}=s, \quad \hat{r}=\bar{x}-s \tag{2.10}$$

Here , $\bar{x}$ is the sample mean , and s is the sample standard deviation [3].

(3) Modified Moment Estimators (MME)

Modified Moment Estimators[3] are variations of MLE and ME that employ the first-order statistic $X_{(1)}$, usually replacing the ME equation involving the third moment by one comparing the smallest observation with $X_{(1)}$

$$E(X_{(1)})=r+\frac{\theta}{n} \tag{2.11}$$

and the MME equation are

$$\bar{x}=r+\theta , \quad x_{(1)}=r+\frac{\theta}{n} \tag{2.12}$$

from which we get

$$\hat{\theta}=\frac{n(\bar{x}-x_{(1)})}{n-1}\ ,\qquad \hat{\gamma}=\frac{nx_{(1)}-\bar{x}}{n-1} \tag{2.13}$$

Ⅲ. Process Capability Measures

3.1 process capability index based on exponential distribution curvers (I_e)

The process capability index over the Weibull distribution process by Mukherjee[4] is given by

$$I=\frac{U-L}{R}=\frac{U-L}{F^{-1}(p_2)-F^{-1}(p_1)}$$

$$=\frac{\theta^{-1/k}(U-L)}{[-\log(1-p_2)]^{1/k}-[-\log(1-p_1)]^{1/k}} \tag{3.1}$$

Here , We consider a quality(reliability) characteristic (x) having the exponential probability density function.

$$f(x;\frac{1}{\theta})=\frac{1}{\theta}e^{-\frac{1}{\theta}} \tag{3.2}$$

Where ① U , L = upper and lower specification limits , respectively , for the quality characteristics(x).

② $\frac{1}{\theta}$=scale parameter of the exponential distribution.

③ μ=mean time to failure = θ

④ σ= standard deviation of time to failure = θ

⑤ t_1 , t_2 =lower and upper limites, respectively , of the process capability interval (also called the natural process interval)

⑥ $R=t_2-t_1$

⑦ p_1 , p_2 = areas under the exponential distribution curve to the left of t_1 and t_2 . Thus , t_1 and t_2 are just the quantiles of the exponential distribution of order p_1 and p_2 , respectively.

⑧ $C=p_2-p_1$ = the area to be covered by the capability interval.

Similarly , process capability index based on exponential distribution curves (I_e) may be obtained from the process capability index over the Weibull distribution (I)

$$I=\frac{U-L}{R}=\frac{U-L}{F^{-1}(p_2)-F^{-1}(p_1)}$$

$$=\frac{\frac{1}{\theta}(U-L)}{[-\log(1-p_2)]-[-\log(1-p_1)]} \tag{3.3}$$

As an estimate of the process capability index , I_e works out as

$$\hat{I}_e = \frac{\hat{\frac{1}{\theta}}(U-L)}{[-\log(1-p_1-C)]-[-\log(1-p_1)]}$$
$$= (U-L)G(\hat{\theta}^{-1}) \tag{3.4}$$

Where $\hat{\theta}^{-1}$ is MLE

3.2 Process capability Index based on the Pearson System

The Process capability Index are numerical quantities whose purpose is to indicate to what degree the output of a process is capable of staying within preassigned specifications , the so-called spec limits , when the process is in control. The condition of being in control is indicated on a control chart when all data points lie within certain control limits and no apparent trends or patterns are present.

Process are operated at some target , or nominol , value and have an upper limit *USL* and lower limit *LSL*. The standard process capability indices for normal distribution are

$$C_p = \frac{USL-LSL}{6\sigma} \tag{3.5}$$

and

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} = \min(C_{pl}, C_{pu}) \tag{3.6}$$

Clenents[5] has developed a tecknique for adjusting C_p or C_{pk} non-normal situations based on Pearson curves , the so-called the percentile method is proposed. In this case , the formulas

$$C_p = \frac{USL-LSL}{P_{0.99865}-P_{0.00135}} \tag{3.7}$$

and

$$C_{pk} = \left\{\frac{USL}{P_{0.99865}-x}\ \frac{x-LSL}{x-P_{0.00135}}\right\} = \min(C_{pu}, C_{pl}) \tag{3.8}$$

are usually used , although there are good reasons for replacing $\bar{x}$ here with the median $P_{0.5}$

In these formulas , $P_t = F^{-1}(t)$ is the t th percentile of the true distribution function F. when F in unknown , and assumed non-normal , E_q.(3.7) and (3.8) may be used to compute a reason approximation to F.

3.3 Process Capability Index based on the Johnson System

Johnson provided an alternation to the Pearson systems of curves for modeling non-normal distribution. His approach to start with a small set of curves capable of approximating the shape of wide specturm of probability distribution and then to find simple transformations that would convert these curve into the standard normal , or Z distribution. For S_L curves (lognormal) , the Johnson transfornation can be witten

$$Z = r^* + \eta \ln(x - \varepsilon) \tag{3.9}$$

Farnum[6] is used Johnson curves to desribe non-normal process data

$$\widehat{C_{pk}} = \min\left(\frac{Z_U}{3}, -\frac{Z_L}{3}\right) = \min(C_{pu}, C_{pl}) \tag{3.10}$$

where Z_U Z_L mean the specification limits *USL* and *LSL* values , respectively.

IV. Illustrative Example

To illustrative the use of the Lilliefors test for normality , as gien by Owen and Li[7]lead to exponential data

0.029	0.046	0.133	0.194	0.265	0.287	0.322	0.433	0.441	0.464
0.483	0.528	0.606	0.789	0.940	1.681	1.766	2.014	3.088	3.279

To evaluate process capability , We will use upper specification limit *USL*=3 for these data.

4.1 The Lilliefors Test for the Exponential Distribution

(1) Hypotheses

H_0: The random sample has the exponential distribution

$$F(x) = \begin{cases} 1 - e^{x/t} & , x > 0 \\ 0 & , x < 0 \end{cases} \tag{4.1}$$

Where t is an unknown parameter

H_1: The distribution of X in not exponential

The largest absolute deviation between $S(x)$ and $F^*(x)$ is seen to equal 0.1559.

The null hypothesis of an exponential distribution may be rejected $\alpha = 0.05$ only if T exceeds 0.2345 (n=20, 1- α =0.95).

Since T=0.1559 , the null hypothesis is accepted.

(2) Parameter Estimation

① MLE

Fron E_q.(2.9) , it is given an $\hat{\theta}$=0.8894

② ME

Fron E_q.(2.10) , it is given an $\hat{\theta}$=0.9670

③ MME

Fron E_q.(2.13) , it is given an $\hat{\theta}$=0.9057

4.2 Evaluation of Process Capability Measures

(1) Process Capability index based on exponential distribution curves (I_e)

① MLE

From the table of Mukherjee et al.[8]

for k=1 , the process capability index $I_e = U/3.912\sigma$

Hence , the estimated process capability index is

$$\hat{I}_e = \frac{3}{(3.912)(0.8894)} = 0.86$$

It indicates that the process is nearly capable.

② ME

Similarly , the estimated process capability index is

$$\hat{I}_e = \frac{3}{(3.912)(0.9670)} = 0.79$$

③ MME

Similarly , the estimated process capability index is

$$\hat{I}_e = \frac{3}{(3.912)(0.9057)} = 0.85$$

(2) Process capability indices based on the Pearson system

As computed from example , We get the statistics of sample as follows:

$\hat{\mu} = 0.8894$, median $P_{0.5} = 0.369$, $P_{0.99865} = 3.628$, $P_{0.00135} = 0.153$, $\hat{\sigma} = 0.9670$

Hence , the estimated process capability indices are $\hat{C}_p = 0.86$ and $\hat{C}_{pk} = 0.82$.

It indicates that the process is nearly capable.

(3) Process capability index based on the Johnson system

To choose appropriate Johnson curve , a discriminant function by Slifker and Shapiro[9] applied in this example S_L distribution.

Then , on numerical computation , We get $\hat{Z} = 0.5770 + 0.7973 \ln(x - 0.009)$

Hence , the estimated process capability Index is

$$\hat{C}_{pk} = \frac{Z_u}{3} = \frac{1.45}{3} = 0.48$$

It indicates that the process is very poor.

The evaluation of process capability measures on the above-mentioned system are tabulated in the table 1.

Table 1. The evaluation of process capability measures

Methods of Parameter Estimation \ Process Capability Indices	Population Type			
	Exponential Distribution	Pearson System		Johnson System
	(I_e)	$\hat{C}_p$	$\hat{C}_{pk}$	$\hat{C}_{pk}$
MLE	0.86 *	0.86 *	0.82	0.48
ME	0.79			
MME	0.85 *			
Capability	*nearly capable	*nearly capable		very poor

V. Summary and Conclusions

The main objective of this paper to purpose a evaluating methods of process capability measures for exponential distributed quality characteristics.

For correctly evaluating process capability , the first thing , exponential data is applied the Lilliefors test statistic to the null hypothesis of nornality. The next , exponential parameters is estimated in terms of MLE , ME , MME and then evaluated , respectively , process capability index based on exponential curved (I_e) proposed by in this study and process capability indices based on Pearson system and Johnson system.

Form calculated results in the table 1 , it makes little difference the MLE , MME method and Pearson system. These vlaue indicates that the process in nearly capable. The ME method is not good in this example. With sample data , a suppose case , We will be accept the lognormal in favor of process capability for lognormal distribution is estimated $\hat{C}_{pk} = 0.48$, namely , Johnson system is underestimated than the others.

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저 자 소 개

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