A Test For Trend Change in Failure Rate Using Censored Data

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ABSTRACT

The problem of trend change in the failure rate is great interest in the reliability and survival analysis. In this paper we develop a test statistic for testing whether or not the failure rate changes its trend using random censored data. The asymptotic normality of the test statistic is established. We discuss the efficiency values of loss due to censoring.

1. Introduction

Reliabilists find it useful to categorize life distributions(distributions such that $F(x) = 0$ for $x < 0$ according to different aging properties. These categories are useful for modeling situations where items improve or deteriorate with age. If F has a density f, the failure rate is defined as

$$r(x) = \frac{f(x)}{\bar{F}(x)},$$

where $\bar{F}(x) = 1 - F(x)$ is the reliability function.

Based on the behavior of failure rate, various nonparametric classes of life distributions have been defined. One such class of distributions is called as the bathtub-shaped failure rate (BTR), if there exists a change point τ such that $r(t)$ is decreasing in $[0, \tau)$ and increasing in $[\tau, \infty)$. The dual class is "upside-down bathtub-shaped failure rate (UBR). If $r(t)$ is constant for all $t \geq 0$, it is said that the life distribution F is constant failure rate (CFR). See Guess and Proschan (1988) and the references therein for examples and applications of the BTR(UBR) class.

It is well known that F is CFR if and only if F is an exponential distribution (i.e., $F(t) = \exp(-t/\mu)$ for $t \geq 0$, $\mu > 0$). Due to this "no-aging" property of the exponential distribution, it is of practical interest to know whether a given life distribution F is CFR or BTR. Therefore, we consider the problem of testing

H_0 : F is CFR

against

H_1 : F is BTR (not CFR).

When the dual model is proposed, we test H_0 against

H_1' : F is UBR (not CFR).

Matthews and Farewell (1982) and Matthews, Farewell and Pyke (1985) considered the problem of testing for a CFR against the alternative with two constant failure rates involving a single change-point. Park (1988) proposed a test for CFR versus BTR (UBR), assuming that the proportion of the population that fails at or before the change-point of failure rate is known.

The trend change in mean residual life has been discussed by Guess, Hollander and Proschan (1986), Aly (1990), Hawkins, Kochar and Loader (1992), Lim and Park (1998), and Na (1998).

In this paper we develop a test statistic for testing exponentiality against BTR (UBR) alternative using censored data. We assume that the change-point is known. We derive the asymptotic null distribution of our test statistic. To establish the asymptotic distribution of our test statistic, we used the technique of Joe and Proschan (1982). We discuss the efficiency values of loss due to censoring.

Section 2 is devoted to develop a test statistic for testing exponentiality against BTR(UBR) alternative. The efficiency values of loss due to censoring are presented in Section 3.

2. Test for Trend Change in Failure Rate

In this section we develope a test statistic for testing exponentiality against BTR(UBR) alternative. We assume that the change point of failure rate is known or has been specified by the user. Motivated by Park (1988), we consider the parameter

$$T(F) \quad = \quad \int_0^{\tau} \int_0^t [r(s) - r(t)]\bar{F}(s)\bar{F}(t) \; dsdt + \int_{\tau}^{\infty} \int_{\tau}^t [r(t) - r(s)]\bar{F}(s)\bar{F}(t) \; dsdt$$

as a measure of the deviation from H_0 in favor of H_1. Straight calculations show that $T(F)$ can be rewritten as

$$
\begin{aligned}
T(F) &= \int_0^\tau \{(2-F(\tau))\bar{F}(t) - 2\bar{F}^2(t)\}dt + \int_\tau^\infty \{(F(\tau)-1)\bar{F}(t) + 2\bar{F}^2(t)\}dt \\
&= \int_0^\infty B(F(x), F(\tau))dx, \qquad (2.1)
\end{aligned}
$$

where

$$
B(u,v) \equiv \begin{cases} B_1(u,v) \equiv -2(1-u)^2 + (2-v)(1-u) & \text{if } 0 \le u \le v, \\ B_2(u,v) \equiv 2(1-u)^2 - (1-v)(1-u) & \text{if } v < u \le 1. \end{cases}
$$

In our randomly censored model, we replace F in (2.1) by the Kaplan-Meier(KM) estimator defined in (2.2) below.

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed(i.i.d.) according to a continuous life distribution function F and let $C_1, C_2, \cdots, C_n$ be i.i.d. according to a continuous life distribution G where C_i is the censoring time associated with T_i, $i = 1, 2, \cdots, n$. In random censoring case we can only observe $(Y_1, \delta_1), \cdots, (Y_n, \delta_n)$ where

$$
Y_i = \min(X_i, C_i), \quad \delta_i = I(X_i \le C_i), 1 \le i \le n.
$$

It is assumed that X's and Y's are mutually independent. The random variable Y_i is said to be uncensored or censored according as $\delta_i = 1$ or $\delta_i = 0$. Therefore $Y_1, \cdots, Y_n$ are observations from a life distribution H with reliability function $\bar{H} = \bar{F}\bar{G} = (1-F)(1-G)$. The Kaplan-Meier estimator is defined by

$$
\bar{F}_n(x) = 1 - F_n(x) = \prod_{\{i: X_{(i)} \le x\}} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}}, \qquad (2.2)
$$

where $Y_{(1)} \le \cdots \le Y_{(n)}$ are the ordered Y's and $\delta_{(i)}$ is the censoring status corresponding to $Y_{(i)}$. We treat $Y_{(n)}$ as uncensored observation whether it is uncensored or not. When censored observations are tied with uncensored we treat the uncensored as preceeding the censored.

As to the problem of testing H_0 against H_1, we propose a test statistic

$$
T_n^c = \frac{\sqrt{n}T(F_n)}{\hat{\mu}_F}
$$

where

$$
\hat{\mu}_F = \sum_{i=1}^n \left\{ \prod_{v=1}^{i-1} \left(\frac{n-v}{n-v+1}\right)^{\delta_{(v)}} \right\} (Y_{(i)} - Y_{(i-1)}).
$$

For computational purpose, $T(F_n)$ may be written as

$$\begin{aligned} T(F_n) &= \sum_{i=1}^{i^*} B_1\Big(\prod_{v=1}^{i-1} c_v^{\delta_{(v)}}\Big)(Y_{(i)} - Y_{(i-1)}) + B_1\Big(\prod_{v=1}^{i^*} c_v^{\delta_{(v)}}\Big)(\tau - Y_{(i^*)}) \\ &+ B_2\Big(\prod_{v=1}^{i^*} c_v^{\delta_{(v)}}\Big)(Y_{(i^*+1)} - \tau) + \sum_{i=i^*+2}^{n} B_2\Big(\prod_{v=1}^{i-1} c_v^{\delta_{(v)}}\Big)(Y_{(i)} - Y_{(i-1)}), \end{aligned}$$

where $c_v = (n-v)/(n-v+1)$, $B_1(u) = (2 - F_n(\tau))u - 2u^2$, $B_2(u) = (F_n(\tau) - 1)u + 2u^2$, and $0 = Y_{(0)} < Y_{(1)} < \cdots < Y_{(i^*)} \leq \tau < Y_{(i^*+1)} < \cdots < Y_{(n)}$.

When there is no censoring this test statistic reduces to the one T_n^* which is obtained by replacing F in (2.1) with empirical distribution.

To establish asymptotic normality of T_n^c, we assume the following conditions on the distributions F and G.

$$\text{(i)} \quad \int_0^\infty \bar{F}^\beta(x)dx < \infty \quad \text{and} \quad \int_0^\infty \{\bar{F}^{2\beta}(x)\bar{G}(x)\}^{-1}dF(x) < \infty,$$

for some $\beta \in (0, 1/2)$, and

$$\text{(ii)} \quad \sqrt{n}\int_{Y_{(n)}}^\infty \bar{F}(x)dx \xrightarrow{p} 0.$$

The derivation of the asymptotic normality of T_n^c is similar to that of Guess(1984), using the techniques of Joe and Proschan(1982) and Gill(1983). The asymptotic distribution of T_n^c is summarized in Theorem 2.1.

THEOREM 2.1 Suppose F and G are continuous distributions. Assume that F' exists at τ and $F'(\tau)$ is positive. If conditions (i) and (ii) above are satisfied, then

$$\begin{aligned} &\sqrt{n}(T(F_n) - T(F)) \xrightarrow{d} -\int_0^\infty J(F(t), F(\tau))V(t)dt \\ &-\left(\int_0^\tau J_1(F(t), F(\tau))dt \int_\tau^\infty J_2(F(t), F(\tau))dt\right) V(\tau) \end{aligned} \tag{2.3}$$

where $J(u,v) = -\partial B(u,v)/\partial u$, $J_1(u,v) = -\partial B_1(u,v)/\partial v$, $J_2(u,v) = -\partial B_2(u,v)/\partial v$ and $V(t)$ denotes a mean zero Gaussian process with covariance

$$\bar{F}(x)\bar{F}(y)\int_0^{x\wedge y} \frac{dF}{\bar{F}^2\bar{G}}.$$

Under H_0, i.e. F is exponential with mean μ,

$$T_n^c \xrightarrow{d} N[0,\ \sigma^2] \quad \text{as } n \to \infty,$$

where

$$\sigma^2 = \int_0^1 \frac{4z^3 - 4(1-p)z^2 + (1-p)^2 z}{\bar{H}(-\mu \log z)} dz + \int_{1-p}^1 \frac{(3-2p)z - 4z^2}{\bar{H}(-\mu \log z)} dz, \tag{2.4}$$

and $p = 1 - \exp(-\tau/\mu)$.

Since the asymptotic null variance σ^2 depends on the nuisanse parameters μ and H, we need a consistent estimator of σ^2. We can obtain a consistent estimator of σ^2, σ_n^2, by replacing $\bar{H}$ in (2.3) with $\bar{H}_n$, the empirical reliability function of $Y_1, \cdots, Y_n$. For computational purpose, we have

$$\begin{aligned}
\sigma_n^2 &= \frac{1}{2}\hat{p}^2 - \frac{2}{3}\hat{p} + \frac{1}{3} \\
&+ \sum_{i=1}^{n-1} \frac{n}{(n-i+1)(n-i)} \left(B_i(4) - \frac{4}{3}\bar{\hat{p}} B_i(3) + \frac{1}{2}\bar{\hat{p}}^2 B_i(2) \right) - n \left(B_n(4) - \frac{4}{3}\bar{\hat{p}} B_n(3) + \frac{1}{2}\bar{\hat{p}}^2 B_n(2) \right) \\
&+ \sum_{i=1}^{k} \frac{n}{(n-i+1)(n-i)} \left(\frac{3-2p}{2} B_i(2) - \frac{4}{3} B_i(3) \right) - \frac{n}{n-k} \frac{(2\hat{p}+1)(1-\hat{p})^2}{6},
\end{aligned}$$

where $\hat{p} = 1 - \exp(-\tau/\hat{\mu}_F)$, $B_i(a) = \exp(-aY_{(i)}/\hat{\mu}_F)$ and $Y_{(k)} \le -\hat{\mu}_F \log \bar{\hat{p}} < Y_{(k+1)}$.

The BTR test procedure rejects H_0 in favor of H_1 at the approximate level α if $T_n^c/\sigma_n \ge z_\alpha$, where z_α is the upper α-quantile of standard normal distribution. Analogously, the approximate α level test of H_0 versus H_1' reject H_0 if $T_n^c/\sigma_n \le -z_\alpha$.

3. THE EFFICIENCY LOSS DUE TO CENSORING

In this section we study the efficacy loss due to censoring by comparing the efficiency of Park (1988) test based on T_n^* for uncensored model with the efficacy of our BTR test based on T_n^c for randomly censored model. Since T_n^c and T_n^* have the same asymptotic means we get the Pitman ARE of the test based on T_n^c relative to that based on T_n^* as $ARE_F(T_n^c, T_n^*) = \sigma_0^2/\sigma^2$ where σ_0^2 is the asymptotic null variance of $\sqrt{n}T_n^*$. If in particular the censoring distribution is exponential, $\bar{G}(x) = \exp(-\rho x)$ for $x \ge 0$ with $\rho < 1$. Then we get

$$ARE_F(T_n^c, T_n^*) = \left(p^2 - p + \frac{1}{3}\right) \Big/ \left(\frac{(p+1)^2\rho^2 - (5p+1)(p+1)\rho + 6p^2 + 2}{(3-\rho)(2-\rho)(1-\rho)} \right.$$

Table 3.1

The Pitman ARE of T_n^c relative T_n^* for some τ and amount of censoring

	ρ			
τ	1/9	1/4	3/7	2/3
0.1	0.841	0.647	0.429	0.181
0.7	0.867	0.695	0.473	0.212
2.3	0.903	0.786	0.644	0.475

$$+ \quad \left. \frac{(2p+1+\bar{p}^{1-\rho}-2\bar{p}^{2-\rho})\rho+2-4p-2\bar{p}^{1-\rho}}{(2-\rho)(1-\rho)} \right), \tag{3.1}$$

where $p = 1 - \exp(-\tau)$.

Table 3.1 indicates $ARE_F(T_n^c, T_n^*)$ for some different amount of censoring, where $\bar{G}(x) = \exp(-\rho x)$ when $\tau = 0.1, 0.7, 2.3$. ¿From Table 3.1 we notice that the value of $ARE_F(T_n^c, T_n^*)$ increases to 1 as ρ decreases. As expected from (3.1), it is obvious that the efficiency loss $ARE_F(T_n^c, T_n^*)$ tends to 1 as ρ tends to 0 (corresponding to the case of no censoring). Also we notice that the efficiency loss due to censoring is large when $\tau = 0.7$ and the smallest efficiency loss is obtained when $\tau = 2.3$.

REFERENCES

[1] Aly, E. E. A. A. (1990): "Tests for Monotonicity Properties of the Mean Residual Life Function", *Scandinavian Journal of Statistics*, Vol. 17, No. 3, pp. 189-200.

[2] Gill, R. D. (1983), "Large Sample Behavior of the Product Limit Estimator on the Whole Line", *The Annals of Statistics*, Vol. 11, pp. 49–58.

[3] Guess, F. (1984), *Testing Whether Mean Residual Life Changes Trend*, Ph. D. dissertation, Florida State University.

[4] Guess, F., Hollander, M. and Proschan, F. (1986), "Testing Exponentiality versus a Trend Change in Mean Residual Life", *Annals of Statistics* 14, 1388-1398.

[5] Guess, F., and Proschan, F. (1988), "Mean Residual Life: Theory and Application", *Handbook of Statistics Vol. 7: Quality Control and Reliability*, 215–224, North-Holland, Amsterdam.

[6] Hawkins, D. L., Kochar, S. and Loader, C. (1992), "Testing exponentiality against IDMRL Distributions with Unknown Change Point", *Annals of Statistics*, Vol. 20, 280-290.

[7] Joe, H. and Proschan, F. (1982), Asymptotic Normality of L–statistics with randomly censored data", *Florida State University Technical Report*, M613.

[8] Matthews, D. E. and Farewell, V. T. (1982), "On Testing for a Constant Hazard against a Change-Point Alternative", *Biometrics* 38, 463-468.

[9] Matthews, D. E., Farewell, V. T. and Pyke, R. (1985), "Asymptotic Score-statistic Processes and Tests for Constant Hazard against a Change-Point Alternative", *Annals of Statistics* 13, 583-591.

[10] Lim, J. H., and Park, D. H. (1998). "A Family of Tests for Trend Change in Mean Residual Life", *Communications in Statistics. Theory and Methods*, 27(5), 1163–1179.

[11] Na, M. H. (1998), *A Study on Statistical Inference of Mean Residual Life* , Ph. D. dissertation, Seoul National University, Korea.

[12] Park, D. H. (1988), "Testing Whether Failure Rate Changes its Trend", *IEEE Transactions on Reliability* 37, 375-378.