The Optimal Limit of the Number of Consecutive Minimal Repairs

Jongho Bae* and Eui Yong Lee†

Abstract

Brown and Proschan(1983) introduced a model for imperfect repair. At each failure of a device, with probability p, it is repaired completely or replaced with a new device(perfect repair), and with probability $1-p$, it is returned to the functioning state, but it is only recovered to its condition just prior to failure(imperfect repair or minimal repair). In this paper, we limit the number of consecutive minimal repairs by n. We find some useful properties about μ_k, the expected time between the k-th and the $(k+1)$-st repair under the assumption that only minimal repairs are performed. Then, we assign cost to each repair and find the value of n which minimizes the long-run average cost for a fixed p under the condition that distribution F of the device is DMRL.

1 Introduction

Brown and Proschan(1983) suggested the following model:
A device is repaired at failure. With probability p, it is returned to the 'good-as-new' state (perfect repair), with probability $1-p$, it is returned to the functioning state, but it is only as good as a device of age equal to its age at failure (imperfect repair, or minimal repair). Repair takes negligible time. They called this *failure process*. When we let F be the life distribution of a device, the case $p = 1$ is reduced to a renewal process with inter-arrival time distribution F, and the case $p = 0$ the upper record value process (Shorrock(1972)), a non-homogeneous Poisson process with intensity function coinciding with the failure rate function $r(t)$ of F. For the *failure process*, Lee and Lee(1999) assign cost to each repair and found p minimizing the long-run average cost under the condition that F is DMRL (For the definition of DMRL, see Barlow and Proschan (1965)).

Now, we suggest a modified *failure process* by adding a parameter to the *failure process*. It is possible to perform repeatedly only minimal repairs in the original *failure process*. But, in practice, one may perform a perfect repair once in a while regardless of p, if only minimal repairs are repeated for a while. Hence, we limit the number of consecutive minimal repairs by n ($n = 0, 1, 2, ..., \infty$), i.e., if we have performed minimal repairs n times

*Department of Mathematics, Pohang University of Science and Technology, Pohang, 790-784, Republic of Korea.

†Department of Statistics, Sookmyung Women's University, Seoul, 140-742, Republic of Korea.

successively, we do perform a perfect repair at the next failure regardless of p. The case $n=\infty$ is considered as the *failure process* of Brown and Proschan(1983). For distinction, we call our model $(F,p,n)-$ *failure process* and the model of Brown and Proschan(1983) $(F,p)-$ *failure process*. In the $(F,p,n)-$ *failure process*, the points where the perfect repair is performed also form an embedded renewal process as in the $(F,p)-$ *failure process*.

We let X, F, and r be the life length of a device (or a system), the distribution of X, and the failure rate of F, respectively, and also we let $X_{p,n}$, $F_{p,n}$, and $r_{p,n}$ be the time between two successive perfect repairs, the distribution of $X_{p,n}$, and the failure rate of $F_{p,n}$, respectively. We define $N(t)$ to be the number of failures until time t, and $N_{p,n}$ to be the number of failures until the first perfect repair. Then, we can easily see that $\Pr(N_{p,n}=k)=pq^{k-1}$ for $k=1,2,...,n$ and q^n for $k=n+1$, where $q=1-p$. We assume that the mean of the life length of the device is finite and F is absolutely continuous, i.e., $\int_0^\infty \bar{F}(t)dt<\infty$ and F has a density and a failure rate function. Finally, we denote the upper bound of the support of F by $A=\sup\{t|\bar{F}(t)>0\}$.

2 Distribution of $X_{p,n}$

We obtain the distribution $F_{p,n}(t)$ of $X_{p,n}$ and the failure rate $r_{p,n}(t)$ of $F_{p,n}$ in terms of $F(t)$ and $r(t)$.

Lemma 2.1 *For $0<t<A$ and $k=0,1,2,...,n$,*

$$\Pr(X_{p,n}>t, N(t)=k)=q^k\,\frac{e^{-\int_0^t r(s)\,ds}\,(\int_0^t r(s)\,ds)^k}{k!}.$$

proof Since $(F,p,n)-$ *failure process*, until a perfect repair, can be considered as a non-homogeneous Poisson process $\{M(t);t\geq 0\}$ having intensity function $r(t)$, the event that $X_{p,n}>t$ and $N(t)=k$ is equivalent to the event that $M(t)=k$ and $N_{p,n}>k$, for $k=0,1,2,...,n$. And, since $N_{p,n}$ and $M(t)$ are independent,

$$\begin{aligned}\Pr(X_{p,n}>t,N(t)=k) &= \Pr(N_{p,n}>k)\,\Pr(M(t)=k)\\ &= q^k\,\frac{e^{-\int_0^t r(s)\,ds}\,(\int_0^t r(s)\,ds)^k}{k!}.\end{aligned}$$

Theorem 2.2

$$\bar{F}_{p,n}(t)=\bar{F}(t)\sum_{k=0}^{n}\frac{(-q\ln\bar{F}(t))^k}{k!},\quad 0\leq t<A.$$

proof By applying the law of total probability,

$$\begin{aligned}
\Pr(X_{p,n} > t) &= \sum_{k=0}^{n} \Pr(X_{p,n} > t, N(t) = k) \\
&= \sum_{k=0}^{n} q^k \frac{e^{-\int_0^t r(s)\,ds} \left(\int_0^t r(s)\,ds\right)^k}{k!} \quad \text{by Lemma 2.1} \\
&= \bar{F}(t) \sum_{k=0}^{n} \frac{(-q \ln \bar{F}(t))^k}{k!}.
\end{aligned}$$

Corollary 2.3 *The failure rate of $F_{p,n}$ exists and*

$$r_{p,n}(t) = r(t) \left\{ p + q \frac{(q \int_0^t r(s)\,ds)^n / n!}{\sum_{k=0}^{n} (q \int_0^t r(s)\,ds)^k / k!} \right\}, \quad 0 < t < A.$$

3 The Behavior of μ_k

Consider the upper record vaule process corresponding to F (Shorrock(1972)) (i.e. Non-homogeneous Poisson process with intensity function $r(t)$). In this process, we let X_k be the time between the k-th repair and the $(k+1)$-st repair. We assume that the 0-th repair occurs at time 0. We define μ_k as $E(X_k)$. At first, we calculate μ_k from F.

Proposition 3.1

$$\begin{aligned}
\mu_k &= \int_0^A \frac{\bar{F}(x) (-\ln \bar{F}(x))^k}{k!} \, dx \\
&= \int_0^A \frac{e^{-\int_0^x r(s)\,ds} \left(\int_0^x r(s)\,ds\right)^k}{k!} \, dx,
\end{aligned}$$

proof For $0 \leq x < A$,

$$\begin{aligned}
\Pr(X_0 + X_1 + \cdots + X_k > x) &= \Pr(\text{the number of failures in } (0, x) \leq k) \\
&= \sum_{j=0}^{k} \frac{e^{-\int_0^x r(s)\,ds} \left(\int_0^x r(s)\,ds\right)^j}{j!},
\end{aligned}$$

since the process of failure time is a non-homogeneous Poisson process with intensity function $r(s)$. Hence,

$$\begin{aligned}
E(X_0 + X_1 + \cdots + X_k) &= \int_0^A \Pr(X_0 + X_1 + \cdots + X_k > x) \, dx \\
&= \sum_{j=0}^{k} \int_0^A \frac{e^{-\int_0^x r(s)\,ds} \left(\int_0^x r(s)\,ds\right)^j}{j!} \, dx.
\end{aligned}$$

Since this equality holds for each $k = 0, 1, 2, ...$, we obtain inductively

$$\begin{aligned}
\mu_k = E(X_k) &= \int_0^A \frac{e^{-\int_0^x r(s)\,ds} \left(\int_0^x r(s)\,ds\right)^k}{k!}\,dx \\
&= \int_0^A \frac{\bar{F}(x)\,(-\ln \bar{F}(x))^k}{k!}\,dx
\end{aligned}$$

Remark

From Theorem 2.2 and Proposition 3.1, $E(X_{p,n})$ can be written as

$$E(X_{p,n}) = \sum_{k=0}^{n} \mu_k q^k.$$

Theorem 3.2 *Suppose that F is in DMRL. Then, μ_k is decreasing in k.*

proof Let τ_k be the time of the k-th failure with $\tau_0 = 0$ in the upper record value process corresponding to F. Notice that

$$\tau_k = X_0 + X_1 + \cdots + X_{k-1}.$$

If we define $g(x) = E(X - x | X > x)$, where X has distribution F, then $g(x)$ is decreasing in x by the definition of DMRL. Now, at first,

$$E(X_1) = E(\,g(\tau_1)\,) \le E(\,g(0)\,) = g(0) = E(X_0),$$

and for $k \ge 1$,

$$\begin{aligned}
E(X_{k+1}) &= \int_0^A \int_0^{A-t} E(\,X_{k+1}|X_0 + X_1 + \cdots + X_{k-1} = t, X_k = x\,)\,dF_{X_k|\tau_k=t}(x)\,dF_{\tau_k}(t) \\
&\le \int_0^A \int_0^{A-t} E(\,X_k|X_0 + X_1 + \cdots + X_{k-1} = t\,)\,dF_{X_k|\tau_k=t}(x)\,dF_{\tau_k}(t) \\
&= \int_0^A E(\,X_k|X_0 + X_1 + \cdots + X_{k-1} = t\,)\,dF_{\tau_k}(t) \\
&= E(X_k).
\end{aligned}$$

From now on, we observe the limiting behavior of μ_k as $k \to \infty$.

Lemma 3.3 *When $A = \infty$,*

(1) If $r(s) \le U$ in $(0, \infty)$, $\mu_k \ge \dfrac{1}{U}$ for all $k = 0, 1, 2, ...$

(2) If $r(s) \ge L$ in $(0, \infty)$, $\mu_k \le \dfrac{1}{L}$ for all $k = 0, 1, 2, ...$

proof

(1) For $k = 0$,

$$\mu_0 = E(X_0) \geq E(Y) = \frac{1}{U},$$

where Y is a random variable with failure rate $r_Y(y) \equiv U$. For $k \geq 1$,

$$\mu_k = \int_0^\infty g(t)\, dF_{\tau_k}(t)$$

from the proof of Theorem 3.2. But, $g(t) \geq 1/U$ for every $t > 0$ since the residual life at time t, $X - t|X > t$, has a failure rate $r(x+t) \leq U$. Hence,

$$\mu_k \geq \frac{1}{U} \int_0^\infty dF_{\tau_k}(t) = \frac{1}{U}.$$

(2) Similar to the proof of (1).

Theorem 3.4 *Suppose that* $A = \infty$ *and* $\mu_0 = \int_0^\infty \bar{F}(x)dx < \infty$.
If $\liminf_{x\to\infty} r(x) = L$ *and* $\limsup_{x\to\infty} r(x) = U$, *then*

$$\frac{1}{U} \leq \liminf_{k\to\infty} \mu_k \leq \limsup_{k\to\infty} \mu_k \leq \frac{1}{L}.$$

In particular, if $\lim_{x\to\infty} r(x) = R$, *then*

$$\lim_{k\to\infty} \mu_k = \frac{1}{R}.$$

(U*,* L*, and* R *may be 0 or* $+\infty$*).*

proof

(1) We show that if $\limsup_{x\to\infty} r(x) = U$, then $\liminf_{k\to\infty} \mu_k \geq \frac{1}{U}$.
When $U = \infty$, this is clear. When $U \leq \infty$, for any $\varepsilon > 0$, there exists $\bar{x} \in [0, \infty)$ such that $r(x) \leq U + \varepsilon$ for $x > \bar{x}$. Therefore, $g(x) \geq 1/(U + \varepsilon)$ for $x > \bar{x}$, by Lemma 3.3. Hence,

$$\begin{aligned} \mu_k &= \int_0^{\bar{x}} g(t)\, dF_{\tau_k}(t) + \int_{\bar{x}}^\infty g(t)\, dF_{\tau_k}(t) \\ &\geq \inf\{g(t)|0 \leq t \leq \bar{x}\}\, F_{\tau_k}(\bar{x}) + \frac{1}{U+\varepsilon} \bar{F}_{\tau_k}(\bar{x}). \end{aligned}$$

Note that $\inf\{g(t)|0 \leq t \leq \bar{x}\}$ is finite since $g(0) = \mu_0 < \infty$. And we know, from the proof of Proposition 3.1, that

$$\bar{F}_{\tau_k}(x) = \sum_{j=0}^{k-1} \frac{e^{-\int_0^x r(s)\,ds} \left(\int_0^x r(s)\, ds\right)^j}{j!} \to 1 \text{ as } k \to \infty,$$

provided that $\int_0^x r(s)ds < \infty$, i.e., $\bar{F}(x) > 0$. Therefore,

$$\liminf_{k\to\infty} \mu_k \geq \frac{1}{U+\varepsilon}.$$

Hence,

$$\liminf_{k\to\infty} \mu_k \geq \frac{1}{U}.$$

(2) We show that if $\liminf_{x\to\infty} r(x) = L$, then $\limsup_{k\to\infty} \mu_k \leq \frac{1}{L}$.

When $L = 0$, this is clear. When $0 < L < \infty$, for any $\varepsilon > 0$, there exists $\bar{x} \in [0, \infty)$ such that $r(x) \geq L - \varepsilon$ for $x > \bar{x}$. Therefore, $g(x) \leq 1/(L-\varepsilon)$ for $x > \bar{x}$. Hence,

$$\begin{aligned} \mu_k &= \int_0^{\bar{x}} g(t)\, dF_{\tau_k}(t) + \int_{\bar{x}}^{\infty} g(t)\, dF_{\tau_k}(t) \\ &\leq \sup\{g(t) | 0 \leq t \leq \bar{x}\}\, F_{\tau_k}(\bar{x}) + \frac{1}{L-\varepsilon}\, \bar{F}_{\tau_k}(\bar{x}). \end{aligned}$$

Note that

$$g(t) = \frac{\int_t^\infty \bar{F}(s)ds}{\bar{F}(t)} \leq \frac{\int_0^\infty \bar{F}(s)ds}{\bar{F}(\bar{x})} = \frac{\mu}{\bar{F}(\bar{x})} < \infty \quad \text{for } 0 \leq t \leq \bar{x}.$$

Thus,

$$\limsup_{k\to\infty} \mu_k \leq \frac{1}{L-\varepsilon},$$

and hence,

$$\limsup_{k\to\infty} \mu_k \leq \frac{1}{L}.$$

When $L = \infty$, $\lim_{x\to\infty} r(x) = \infty$. Therefore, for any $M > 0$, there exists $\bar{x} \in [0, \infty)$ such that $r(x) > M$ for $x > \bar{x}$. Hence, $g(x) \leq 1/M$ for $x > \bar{x}$, and

$$\mu_k \leq \sup\{g(t) | 0 \leq t \leq \bar{x}\}\, F_{\tau_k}(\bar{x}) + \frac{1}{M}\, \bar{F}_{\tau_k}(\bar{x}).$$

Therefore,

$$\limsup_{k\to\infty} \mu_k \leq \frac{1}{M},$$

and hence,

$$\limsup_{k\to\infty} \mu_k = 0.$$

4 Optimal n Minimizing Average Cost

Throughout this section, we assume that F is DMRL. Notice that the time epochs of perfect repairs form a renewal process. So, we can think the duration from a perfect repair to the next pefect repair as a cycle. Then the cycle has the distribution $F_{p,n}$. Let C_p and C_m be the costs of perfect repair and minimal repair, respectively. Define $C(p,n)$ to be the expected cost during a cycle in the renewal process and $C_{avg}(p,n)$ to be the long-run average cost. By the renewal reward theorem (See Ross (1983)),

$$C_{avg}(p,n) = \frac{C(p,n)}{\mu(p,n)},$$

where we denote $E(X_{p,n})$ by $\mu(p,n)$. And,

$$\begin{aligned} C(p,n) &= E\big[\, C_p + C_m(\text{ number of minimal repairs during a cycle })\big] \\ &= C_p + C_m\, E(N_{p,n} - 1) \\ &= \begin{cases} C_p + C_m \dfrac{q}{p}(1 - q^n) & \text{if } 0 < p < 1 \\ C_p + C_m\, n & \text{if } p = 0. \end{cases} \end{aligned}$$

By simple calculation, it can be seen that for any p $(0 \le p < 1)$.

$$\mu(p, n-1) = \mu(p,n) - q^n\, \mu_n,$$

and

$$C(p, n-1) = C(p,n) - C_m\, q^n.$$

Let $\Delta C_{avg}(p,n) = C_{avg}(p,n) - C_{avg}(p,n-1)$ for $n \ge 1$. Then,

$$\begin{aligned} &\Delta C_{avg}(p,n) \\ =\ & \frac{1}{\mu(p,n)\,\mu(p,n-1)} \Big[C(p,n)\left\{\mu(p,n) - q^n\,\mu_n\right\} - \left\{C(p,n) - C_m\, q^n\right\}\mu(p,n) \Big] \\ =\ & \frac{q^n}{\mu(p,n)\,\mu(p,n-1)} \Big[C_m\,\mu(p,n) - \mu_n\, C(p,n) \Big]. \end{aligned}$$

We denote $C_m\,\mu(p,n) - \mu_n\, C(p,n)$ by $\Delta(p,n)$

Lemma 4.1 $\Delta(p,n) = C_m\,\mu(p,n) - \mu_n\, C(p,n)$ *is increasing in* $n \ge 1$*, provided that* F *is DMRL.*

proof

$$\begin{aligned} \Delta(p,n+1) - \Delta(p,n) &= C_m\left\{\mu(p,n+1) - \mu(p,n)\right\} - \mu_{n+1}\, C(p,n+1) + \mu_n\, C(p,n) \\ &= C_m\,\mu_{n+1}\, q^{n+1} - \mu_{n+1}\left\{C(p,n) + C_m\, q^{n+1}\right\} + \mu_n\, C(p,n) \\ &= C(p,n)\,(\mu_n - \mu_{n+1}) \\ &\ge 0, \quad \text{since } F \text{ is DMRL.} \end{aligned}$$

Theorem 4.2 *Suppose that F is DMRL. Let p be a fixed probability of perfect repair ($0 \leq p < 1$). Then, we can determine n so that $C_{avg}(p,n)$ is minimized. (It is possible that $C_{avg}(p,n)$ is minimized at $n = \infty$).*

proof Note that

$$C_{avg}(p, n-1) \geq C_{avg}(p,n) \iff \Delta C_{avg}(p,n) \leq 0 \iff \Delta(p,n) \leq 0.$$

Since $\Delta(p,n)$ is increasing in n, we can determine n ($0 \leq n \leq \infty$), by investigating $\Delta(p,1)$ and $\lim_{n\to\infty} \Delta(p,n)$, so that $C_{avg}(p,n)$ is minimized. Note that

$$\Delta(p,1) = C_m \mu_0 - C_p \mu_1 \text{ for any } p \in [0,1).$$

Hence, we can conclude as follows:

(1) If $C_p < \frac{\mu_0}{\mu_1} C_m$, then $C_{avg}(p,n)$ is minimized at $n = 0$. This means that no permission of minimal repairs is the optimal policy.

(2) If $C_p \geq \frac{\mu_0}{\mu_1} C_m$ and $\lim_{n\to\infty} \Delta(p,n) \leq 0$, then $C_{avg}(p,n)$ is decreasing in n. This means ,if $p > 0$, that $(F,p) - failure\ process$ (i.e. the policy with no limitation) is better than $(F,p,n) - failure\ process$ for any integer n.

(3) If $C_p \geq \frac{\mu_0}{\mu_1} C_m$ and $\lim_{n\to\infty} \Delta(p,n) > 0$, then $C_{avg}(p,n)$ is minimized at the maximum integer n such that $\Delta(p,n) \leq 0$. And n is finite.

Remark

By some algebras, we can observe that $\mu_n \to 0$ implies $\lim_{n\to\infty} \Delta(p,n) > 0$ for $0 \leq p < 1$. In other words, if $A < \infty$ or if $A = \infty$ and $\lim_{x\to\infty} r(x) = \infty$, then $\lim_{n\to\infty} \Delta(p,n) > 0$.

References

[1] Barlow, R. E. and Proschan, F. (1965) *Mathematical Theory of Reliability.* Wiley, New York.

[2] Brown, M. and Proschan, F. (1983) Imperfect repair. *J. Appl. Prob.*, **20**, 851–859.

[3] Lee, E. Y. and Lee, J. (1999) An optimal proportion of perfect repair. *Operation Research Letters*, **25**, 147–148.

[4] Ross, S. M. (1983) *Stochastic processes.* John Wiley, New York.

[5] Shorrock, R. W. (1972) A limit theorem for inter-record times. *J. Appl. Prob.*, **9**, 219–223.