Undecimated 웨이블릿 변환응용

An Application of the Undecimated Discrete Wavelet Transform

이 창 수, 유 경 렬
한양대학교 전기공학과

Chang Soo Lee, Kyung Yul Yoo
Department of Electrical Engineering, Hanyang University
yoo@emc.hanyang.ac.kr

Abstract

This paper introduces a new structure for the undecimated discrete wavelet transform (UDWT). This structure combines the stationary wavelet transform with a lifting scheme and its design is based on a polyphase structure, where the downsampling and split stage are removed. The suggested structure inherits the simplicity of the lifting scheme, such that the inverse transform is easily implemented. The performanace of the proposed undecimated lifting is verified on a signal denoising application.

1. Introduction

The discrete wavelet transform (DWT), based on the octave band tree structure, decomposes an input signal using a set of low and high pass filters followed by a decimator. The traditional DWT has exerted a remarkable influence on several signal processing applications such as denoising, estimation, and compression. However, in signal denoising, the DWT is known to create artifacts around the discontinuities of the input signal. These artifacts degrade the performance of the threshold-based denoising algorithm. It has been shown that many of the artifacts could be suppressed by a redundant representation of the signal [1].

The undecimated discrete wavelet transform (UDWT) differs from the traditional DWT because it does not employ a decimator after filtering. This is also known as the redundant or translation invariant DWT. There are several implementations of the UDWT, such as the stationary and the translation invariant (cycle-spinning) wavelet transforms [1, 2]. The absence of a decimator leads to a redundant input signal representation. This makes a denser approximation to the continuous wavelet transform than that of the DWT. The translation invariant property of the UDWT makes it preferable for use in various signal processing applications, as it relies heavily on spatial information [3].

In this paper, we propose a new structure for the UDWT, based on the lifting scheme. This structure can be described as the upsampled predict and update steps without the split stage. The new structure maintains the same performance as other UDWTs, yet it inherits some benefits through the use of the lifting scheme. These benefits are the integer-to-integer transform and the design simplicity of the inverse transforms.

2. The lifting scheme

Lifting has been developed as a flexible tool used for constructing second generation wavelets in complex geometries and/or irregular sampling. It also provides an efficient method for the construction of biorthogonal wavelets. The inverse wavelet transform is easily computed by reversing or undoing the steps of the forward transform. This leads to an efficient overall implementation of the wavelet transform. Moreover, every FIR wavelet or filter bank can be decomposed into lifting steps in the polyphase form [4, 5].

Lifting, a space-domain construction of biorthogonal wavelets, consists of the iteration of the following three basic operations

Split: Divide the original data into two disjoint subsets. For example, we will split the original data set $x[n]$ into $x_e[n]=x[2n]$, the even indexed points, and $x_o[n]=x[2n+1]$, the odd indexed points.

Predict: Generate the wavelet coefficients $d[n]$ as the error in predicting $x_o[n]$ from $x_e[n]$ using prediction operator P:

$$d[n]=x_o[n]-P(x_e[n])$$

Update: Combine $x_e[n]$ and $d[n]$ to obtain scaling coefficients $c[n]$ that represent a coarse approximation to the original signal $x[n]$. This is accomplished by applying an update operator U to the wavelet coefficients and adding to $x_e[n]$:

$$c[n] = x_e[n] + U(d[n])$$

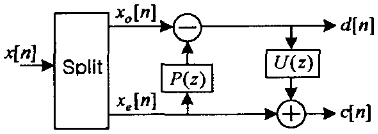

Figure 1. Lifting stage: Split, Predict, Update

3. Design of an undecimated lifting

With the traditional (orthogonal, maximally-decimated) denoising, we suffer from visual artifacts; pseudo-Gibbs oscillations in the neighborhood of discontinuities are caused by the lack of translation invariance of the wavelet basis. When using Haar wavelets, a discontinuity precisely at location $n/2$ will lead to essentially no pseudo-Gibbs oscillations; a discontinuity near a binary irrational like $n/3$ will lead to significant pseudo-Gibbs oscillations. To suppress these artifacts, we've chosen the translation-invariant denoising [1].

Let $X = Wx$ be the (orthogonal) DWT of x and S_R be a matrix performing a circular right shift by R with $R \in Z$. Then

$$X_s = Wx_s = WS_R x = WS_R W^{-1} X \quad (1)$$

which establishes the connection between the wavelet transforms of two shifted versions of a signal, x and x_s, by the orthogonal matrix $WS_R W^{-1}$. Using these transforms, all circular shifts of the input signal are calculated and the denoised output signals are averaged in the reconstruction [1].

To develop an undecimated or translation-invariant lifting scheme, the processing of the stationary wavelet transform is converted into lifting steps. First, the DWT is represented in the polyphase form of the lifting scheme and then the decimator and upsampler are removed. We upsample the lifting operators during the progressive stages of processing. Fig. 2 shows the polyphase implementation of the lifting scheme. The predict and update stages are written by the polyphase matrix $\mathbf{E}(z) = \mathbf{U}(z)\,\mathbf{P}(z)$,

$$\begin{aligned} \mathbf{E}(z) &= \mathbf{U}(z)\,\mathbf{P}(z) \\ &= \begin{bmatrix} 1 & U(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-P(z)U(z) & U(z) \\ -P(z) & 1 \end{bmatrix} \end{aligned} \quad (2)$$

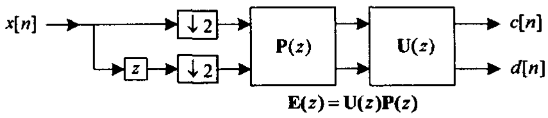

Figure 2. Lifting scheme in the polyphase implementation

where $P(z)$ and $U(z)$ are the predict and update operators respectively. In the case of a 4-point predict and update, $P(z)$ and $U(z)$ are given by

$$P(z) = p_1 z^{-1} + p_2 z^0 + p_3 z^1 + p_4 z^2 \quad (3)$$

$$U(z) = u_1 z^{-2} + u_2 z^{-1} + u_3 z^0 + u_4 z^1 \quad (4)$$

Using noble identities, the lifting steps are rearranged as in Fig. 3.

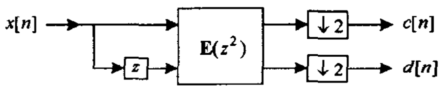

Figure 3. Rearranged polyphase representation

Now, we have:

$$\begin{aligned} \mathbf{E}(z^2) &= \begin{bmatrix} 1-P(z^2)U(z^2) & U(z^2) \\ -P(z^2) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & U(z^2) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P(z^2) & 1 \end{bmatrix} \\ &= \mathbf{U}(z^2)\,\mathbf{P}(z^2) \end{aligned} \quad (5)$$

Removal of the decimators results in undecimated lifting. In practice we do not actually advance the input data $x[n]$. To put z into a polyphase matrix $\mathbf{E}(z^2)$, we add z^{-1} after the polyphase matrix. Finally, we can combine the matrices and redraw the system in Fig. 3. The undecimated analysis matrix $\mathbf{E}_u(z)$ is:

$$\begin{aligned} \mathbf{E}_u(z) &= \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1-P(z^2)U(z^2) & U(z^2) \\ -P(z^2) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \\ &= \begin{bmatrix} 1-P(z^2)U(z^2) & zU(z^2) \\ -z^{-1}P(z^2) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & zU(z^2) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z^{-1}P(z^2) & 1 \end{bmatrix} \end{aligned} \quad (6)$$

For perfect reconstruction, z is inserted before the synthesis polyphase matrix $\mathbf{R}(z^2)$ and z^{-1}after the maxtrix $\mathbf{R}(z^2)$ disappears.

$$\begin{aligned}\mathbf{R}_u(z) &= \begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}\begin{bmatrix}1 & -U(z^2)\\ P(z^2) & 1-P(z^2)U(z^2)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix}\\ &= \begin{bmatrix}1 & -zU(z^2)\\ z^{-1}P(z^2) & 1-P(z^2)U(z^2)\end{bmatrix} \\ &= \begin{bmatrix}1 & -zU(z^2)\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ z^{-1}P(z^2) & 1\end{bmatrix}\end{aligned} \quad (7)$$

From equations (6) and (7), we can verify that the condition of perfect reconstruction is satisfied.

$$\mathbf{R}_u(z)\ \mathbf{E}_u(z) = \mathbf{I} \quad (8)$$

The undecimated 4-point predict operator $P_u(z)$ and update operator $U_u(z)$ are:

$$P_u(z) = z^{-1}P(z^2) = p_1z^{-3} + p_2z^{-1} + p_3z^1 + p_4z^3 \quad (9)$$

$$U_u(z) = zU(z^2) = u_1z^{-3} + u_2z^{-1} + u_3z^1 + u_4z^3 \quad (10)$$

Figs. 4 and 5 show the iterations of the undecimated lifting which construct the forward and inverse UDWT. For convenience, the normalization of the scaling and wavelet coefficients is omitted. To apply the inverse UDWT, we simply undo the forward lifting steps and divide the reconstructed signal by two for energy normalization.

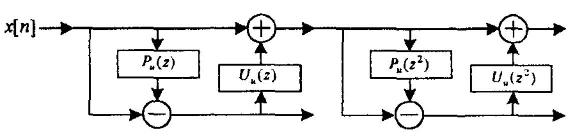

Figure 4. Structure of the forward undecimated lifting

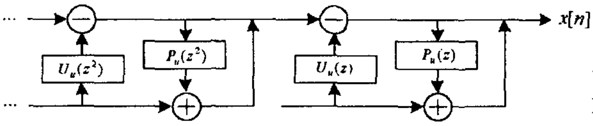

Figure 5. Structure of the inverse undecimated lifting

4. Wavelet denoising

4.1. Wavelet Shrinkage

Wavelet shrinkage, as defined by Donoho [6], refers to a denoising technique consisting of a wavelet transformation of the noisy signal, followed by a shrinking of the small wavelet coefficients to zero while leaving the large coefficients unaffected. Finally, an inverse transformation is performed to acquire the estimated signal. This technique is effective in removing Gaussian noise. The energy compaction property of the wavelet transform tends to concentrate the signal energy into a relatively small number of large coefficients, and the problem of removing Gaussian noise simplifies to extracting the few coefficients of the useful signal that are significantly large.

4.2. Wavelet Domain Empirical Wiener Filtering

However, for any given signal, the MSE-optimal processing is achieved by the Wiener filter which delivers substantially improved performance. In [7] is developed an algorithm for wavelet denoising that uses a wavelet shrinkage estimate as a means to design a wavelet-domain Wiener filter. The algorithm uses two different wavelet bases, one for computing the estimate and the other to implement the Wiener filter. The Wiener filter's coefficients, $h_w(i)$, are computed as:

$$h_w(i) = \frac{w_e^2(i)}{w_e^2 + \hat{\sigma}^2} \quad (11)$$

where $w_e(i)$ represents the wavelet coefficients of the wavelet shrinkage estimate.

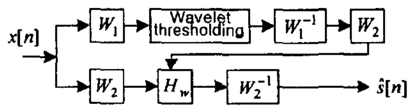

Figure 6. Wavelet-domain empirical Wiener filter

5. Simulations

A series of signal denoising simulations are provided to verify the performance of the proposed undecimated lifting. Donoho's test signals are corrupted by white Gaussian noise (standard deviation of 2) and the signals are rescaled so that the standard deviation is 10. Hard threshold-based denoising (db8) and the empirical Wiener filter (W_1 : db4, W_2 : db8) are then applied to the DWT, UDWT, and undecimated lifting with 4-point (W_1 : 2-point, W_2 : 4-point) predict and update operators.

Tables 1-4 show the MSE results of denoising 4 signals based on 9 different algorithms. The simulation results show that the undecimated lifting leads to the same results as the Donoho's UDWT.

Table 1. Blocks signal

Transform	Estimator	MSE
DWT	Hard threshold	1.801190
	Empirical Wiener	0.701631
	Ideal Wiener	0.614485
UDWT	Hard threshold	0.790288
	Empirical Wiener	0.571506
	Ideal Wiener	0.488356
Undecimated Lifting	Hard threshold	0.641246
	Empirical Wiener	0.547932
	Ideal Wiener	0.398759

Table 2. Bumps signal

Transform	Estimator	MSE
DWT	Hard threshold	0.983686
	Empirical Wiener	0.668808
	Ideal Wiener	0.449471
UDWT	Hard threshold	0.584863
	Empirical Wiener	0.526534
	Ideal Wiener	0.381648
Undecimated Lifting	Hard threshold	0.575987
	Empirical Wiener	0.513663
	Ideal Wiener	0.364545

Table 3. Heavisine signal

Transform	Estimator	MSE
DWT	Hard threshold	0.424377
	Empirical Wiener	0.266778
	Ideal Wiener	0.200360
UDWT	Hard threshold	0.323468
	Empirical Wiener	0.279084
	Ideal Wiener	0.179480
Undecimated Lifting	Hard threshold	0.303638
	Empirical Wiener	0.272126
	Ideal Wiener	0.175064

Table 4. Doppler signal

Transform	Estimator	MSE
DWT	Hard threshold	1.003690
	Empirical Wiener	0.636168
	Ideal Wiener	0.349193
UDWT	Hard threshold	0.506817
	Empirical Wiener	0.457991
	Ideal Wiener	0.282942
Undecimated Lifting	Hard threshold	0.415673
	Empirical Wiener	0.390641
	Ideal Wiener	0.259437

6. Conclusion

We have presented a new structure for the UDWT. It combines the stationary wavelet transform and the lifting scheme to generate a lifting version of the UDWT. This is accomplished by removing the downsampling and split stage of the lifting. For further simplification we inserted the unit delay after the analysis polyphase matrix and we inserted the unit advance before the synthesis polyphase matrix. Using different denoising algorithms, it's proved that the undecimated lifting has the same performance as the other UDWT. In addition, the proposed structure inherits the lifting scheme's simplicity in the design of the inverse transform.

Since the undecimated lifting makes use of the lifting scheme, the applications of the undecimated lifting could be extended to situations with irregular sampling and/or complex geometry.

References:

[1] R. R. Coifman and D. Donoho, Translation-invariant de-noising, in *Wavelets and Statistics*, vol.103 of *Lecture Notes in Statistics*, pp.125-150, Springer-Verlag, New York, 1995.

[2] G. P. Nason and B. W. Silverman, Stationary wavelet transform and some statistical applications, Technical Report, Department of Mathematics, University of Bristol, Bristol, U.K., Feb. 1995.

[3] C. S. Burrus, R. A. Gopinath, and H. Guo, *Introduction to wavelets and wavelet transforms*, Prentice Hall, 1998

[4] W. Sweldens, The lifting scheme: A custom-design construction of biorthogonal wavelets, *J. Appl. Comp. Harm. Anal.*, vol.3, no.2, pp.186-200, 1996

[5] I. Daubechies and W. Sweldens, Factoring wavelet transforms into lifting steps, *J. Fourier Anal. Appl.*, vol.4, no.3, pp.247-269, 1998.

[6] D. Donoho, De-noising by soft-thresholding, *IEEE Trans. on Inform. Theory* , vol.41, pp.613-627, 1995.

[7] S. P. Ghael, A. M. Sayeed and R. G. Baraniuk, Improved wavelet denoising via empirical Wiener filtering, *Proceedings of SPIE*, vol. 3169, pp. 389-399, San Diego, July 1997.