An Estimation of Nonlinear Time Series with ARCH errors using ECLMS algorithm

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Abstract : In this paper, a method of estimating second-order Volterra model with ARCH errors is presented. Then we use an ECLMS algorithm for noise canceling of nonlinear time series. The validity of the proposed method is demonstrated for estimating second-order Volterra model with ARCH errors, using computer simulations.

I. Introduction

Traditional econometric models assume a constant one-period forecast variance. To generalize this implausible assumption, a new class of stochastic processes called autoregressive conditional heteroskedasticity (ARCH) processes[1] were introduced in Engle(1982). These are mean zero, serially uncorrelated processes with nonconstant variances conditional on the past, but constant unconditional variances. This type of model has been already proven to be useful in modeling several different economic phenomena[2]. In these papers, maximum likelihood estimation of the linear regression model with ARCH error was discussed. However, it is well known that there exists the nonlinearity in economic time series by empirical research. It is difficult to analyze the nonlinear time series. Estimation method for nonlinear time series is very complicated.

Recently, the correlation least mean squares(CLMS) algorithm[3] and the extended CLMS(ECLMS) algorithm[4] have been proposed to solve the double-talk problem in the echo canceling system. The characteristic of ECLMS algorithm is to utilize the correlation functions of the input signal instead of the input signal itself. The theoretical investigations for the stability bound of step size of the ECLMS algorithm has been obtain and proved. Noise signal is separated from observed signal by ECLMS algorithm. Therefore, the estimation performance is considerably improved. However, ECLMS algorithm is derived, based on the linear filter. Therefore, this algorithm does not use for nonlinear time series. To solve this problem, we propose the ECLMS algorithm for nonlinear economic time series.

Some recent results on the design and implementation of second-order Volterra filters are presented. The Volterra filter is a nonlinear filter with the filter structure of Volterra series. A simple minimum mean-square error solution for the Volterra filter is derived, based on the assumption that the filter input is Gaussian. Using second-order Volterra filters in proposed algorithm, make it possible to consider wide range applications. Some numerical examples are presented to illustrate that the proposed method can work well for estimating nonlinear time series with ARCH errors.

II. ECLMS for Volterra model

The purpose of this pater is to derive second-order Volterra ECLMS algorithm for economic nonlinear time series. We consider the second-order Volterra filter in the form of

$$\begin{aligned}\tilde{y}(t) &= \sum_{j=0}^{N-1} a_j(t)x(t-j) \\ &+ \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} b_{j,k}(t)x(t-j)x(t-k) \quad (1)\end{aligned}$$

where a_j and $b_{j,k}$ are called the linear and quadratic filter coefficients, respectively and N is length of the filter tap coefficients. As the first term of eqn.(1) is linear part, we can easily treat it based on [4]. Therefore, to avoid unnecessary complication in notation, we consider only second term of Volterra filter as follows:

$$y(t) = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} b_{j,k}x(t-j)x(t-k) \qquad (2)$$

The observed signal $z(t)$ is given by

$$z(t) = y(t) + u(t) \qquad (3)$$

where $u(t)$ is ARCH process noise. The ARCH process noise is created by conditional specification method[5] as follows:

$$u(t) = \sqrt{h(t)}\epsilon(t) \qquad (4)$$

where $h(t)$ is described by

$$h(t) = \sigma^2(t) = \delta_0 + \delta_1 u^2(t-1) + \delta_2 u^2(t-2) + \delta_3 u^2(t-3) + \cdots + \delta_p u^2(t-p) \quad (5)$$

and $\epsilon(t)$ is white Gaussian with variance=1. Then, we define the correlation function $C_{zxx}(p,q)$ between the observed signal $z(t)$ and $x(t-p)x(t-q)$ as follows:

$$C_{zxx}(p,q) = E[z(t)x(t-p)x(t-q)] \quad (6)$$

where $E[\cdot]$ denotes the expectation operator. Using the ergodic property eqn.(6) can be rewritten as

$$C_{zxx}(p,q) = \lim_{t\to\infty} \frac{1}{t+1} \sum_{i=0}^{t} z(i)x(i-p)x(i-q) \quad (7)$$

and approximated as follows:

$$C_{zxx}(t,p,q) = \frac{1}{t+1} \sum_{i=0}^{t} z(i)x(i-p)x(i-q) \quad (8)$$

By substituting eqn.(3) into eqn.(8), $C_{zxx}(t,p,q)$ becomes:

$$C_{zxx}(t,p,q) = C_{yxx}(t,p,q) + C_{uxx}(t,p,q) \quad (9)$$

Furthermore, as $u(t)$ and $x(t-p)x(t-q)$ are two independent signals, $C_{uxx}(t,p,q)$ is almost zero. Therefore, $C_{zxx}(t,p,q)$ becomes:

$$\begin{aligned} C_{zxx}(t,p,q) &\simeq C_{yxx}(t,p,q) \\ &= \frac{1}{t+1} \sum_{i=0}^{t} y(i)x(i-p)x(i-q) \\ &= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} b_{j,k} \phi_{xxxx}(t,j,k,p,q) \quad (10) \end{aligned}$$

where

$$\phi_{xxxx}(t,j,k,p,q) = \frac{1}{t+1} \sum_{i=0}^{t} x(i-j)x(i-k)x(i-p)x(i-q)$$

Also, the Gaussian monent facorization theory [6] is given by

$$\begin{aligned} E[x(t-j)x(t-k)x(t-p)x(t-q)] = \\ E[x(t-j)x(t-k)]E[x(t-p)x(t-q)] \\ +E[x(t-j)x(t-q)]E[x(t-k)x(t-p)] \\ +E[x(t-j)x(t-p)]E[x(t-k)x(t-q)] \quad (11) \end{aligned}$$

Therefore, we have the following relation:

$$\begin{aligned} \phi_{xxxx}(t,j,k,p,q) = \phi_{xx}(t,j-k)\phi_{xx}(t,p-q) \\ +\phi_{xx}(t,j-q)\phi_{xx}(t,k-p) \\ +\phi_{xx}(t,j-p)\phi_{xx}(t,k-q) \quad (12) \end{aligned}$$

where

$$\phi_{xx}(t,j-k) = \frac{1}{t+1} \sum_{i=0}^{t} x(i-j)x(i-k)$$

As a result, eqn.(10) is represented by matrix form as folllows:

$$\widehat{\mathbf{C}}_{zxx}(t) = \mathbf{C}_{xxxx}(t)\widehat{\mathbf{B}}(t) \quad (13)$$

where

$$\widehat{\mathbf{B}}(t) = [\widehat{b}_{0,0}(t), \widehat{b}_{0,1}(t), \cdots, \widehat{b}_{N-1,N-1}(t)]^T$$

$$\widehat{\mathbf{C}}_{zxx}(t) = [\widehat{C}_{zxx}(t,0,0), \widehat{C}_{zxx}(t,0,1), \cdots, \widehat{C}_{zxx}(t,N-1,N-1)]^T$$

$$\mathbf{C}_{xxxx}(t) = [\mathbf{\Phi}^T_{xxxx}(t,0,0) \quad \mathbf{\Phi}^T_{xxxx}(t,0,1) \quad \cdots \quad \mathbf{\Phi}^T_{xxxx}(t,N-1,N-1)]^T$$

$$\mathbf{\Phi}_{xxxx}(t,p,q) = [\phi_{xxxx}(t,0,0,p,q), \phi_{xxxx}(t,0,1,p,q), \cdots, \phi_{xxxx}(N-1,N-1,p,q)]$$

On the other hand, as the $x(t)$ is Gaussian with zero-mean value, all odd order moments of $x(t)$ are zero[6].

$$E[x(t-j)x(t-k)x(t-p)] = 0 \quad (14)$$

Therefore, estimation of quadratic filter coefficients is not effected by linear filter coefficients. In the same way, estimation of linear filter coefficients is not effected by quadratic filter coefficients. The linear and quadratic filter cofficients are estimated in parallel.

Next, the estimation error signal vector $\mathbf{e}(t)$ for the correlation function is defined as follows:

$$\mathbf{e}(t) = \mathbf{C}_{zxx}(t) - \mathbf{C}_{xxxx}(t)\widehat{\mathbf{B}}(t) \quad (15)$$

where

$$\mathbf{C}_{zxx}(t) = [C_{zxx}(t,0,0), C_{zxx}(t,0,1), \cdots, C_{zxx}(t,N-1,N-1)]^T$$

$$\mathbf{e}(t) = [e(t,0,0), e(t,0,1), \cdots, e(t,N-1,N-1)]^T$$

Then we define the cost function as follows:

$$J = E[\mathbf{e}^T(t)\mathbf{e}(t)] \quad (16)$$

Based on the LMS algorithm , we can derive Volterra ECLMS algorithm as follows:

$$\widehat{\mathbf{B}}(t+1) = \widehat{\mathbf{B}}(t) + 2\mu \mathbf{C}^T_{xxxx}(t)\mathbf{e}(t) \quad (17)$$

where μ is the step size for the tap coefficients. The adaptation algorithm, which is normalized to the power of the input correlation function to ensure sufficient conditions for convergence, then becomes:

$$\widehat{\mathbf{B}}(t+1) = \widehat{\mathbf{B}}(t) + \frac{2\mu_0}{1+\mathrm{tr}\left[\mathbf{C}_{xxxx}^T(t)\mathbf{C}_{xxxx}(t)\right]}\mathbf{C}_{xxxx}^T(t)\mathbf{e}(t) \quad (18)$$

where tr[·] shows the trace of a matrix and

$$0 < \mu_0 < 1$$

The proposed estimation system is shown in Fig.1.

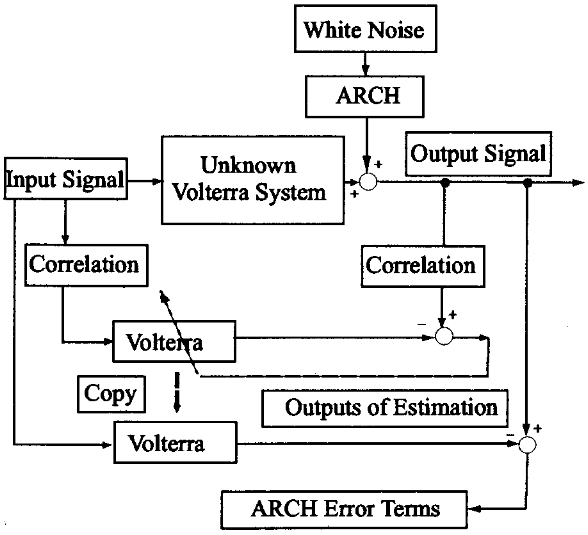

Fig.1. Estimation System.

III. Simulation results

We demonstrate that it is feasible to estimate second-order Volterra model with ARCH error by E-CLMS algorithm. Sample system is assumed as follows:

$$y(t) = \mathbf{A}^T\mathbf{X}(t) + \mathrm{tr}[\mathbf{B}\mathbf{X}(t)\mathbf{X}^T(t)] \quad (19)$$

where

$$\mathbf{A} = \begin{bmatrix} -0.8 & 0.51 & -0.09 \end{bmatrix}^T$$

$$\mathbf{B} = \begin{bmatrix} 0.088566 & -0.045801 & 0.095514 \\ -0.045801 & 0.078566 & -0.055801 \\ 0.095514 & -0.055801 & -0.051434 \end{bmatrix}$$

$$\mathbf{X}(t) = \begin{bmatrix} x(t) & x(t-1) & x(t-2) \end{bmatrix}^T$$

The input signal $x(t)$ is AR process given by

$$x(t) = -0.9x(t-1) + \epsilon(t) \quad (20)$$

The observed noise signal $u(t)$ is ARCH process given by

$$u(t) = \sqrt{h(t)}\epsilon(t) \quad (21)$$

where

$$h(t) = 1 + 0.32u^2(t-1) + 0.12u^2(t-2)$$

Our purpose is to estimate a_j, $b_{j,k}$ and δ_j from observed signal $z(t)$. The output of $\widehat{y}(t)$ esimated by the proposed algorithm and $y(t)$ are shown in Fig.2.

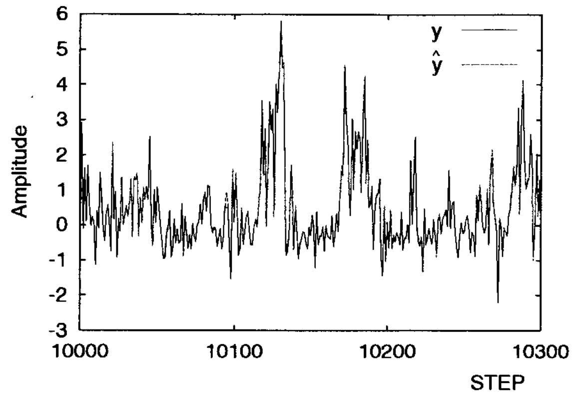

Fig.2. Estimated Signal $\widehat{y}(t)$ and $y(t)$.

To measure the performance of this algorithm, we use $MSE_1(t)$ defined by the following equation:

$$MSE_1(t) = 10\log_{10}\frac{E[\{y(t)-\widehat{y}(t)\}^2]}{E[y^2(t)]} \quad (22)$$

Figure 3 shows the convergence of $MSE_1(t)$ for the ECLMS(length of tap coefficients;10 and 30) and the Volterra ECLMS algorithm.

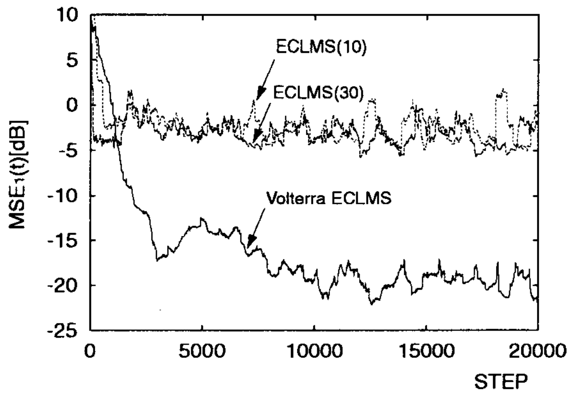

Fig.3. $MSE_1(t)$ of estimated output signal.

ECMLS algorithm (N=10,30)[4] hardly converges at about -3dB. However, the proposed Volterra ECLMS algorithm shows steady convergence at about -20dB. Then we note that the Volterra ECLMS algorithm has good convergence.

To measure the performance of the algorithm, we use $MSE_2(t)$ defined by the following equation:

$$MSE_2(t) = 10\log_{10}\frac{e_a + e_b}{\sum_{i=0}^{2} a_i^2 + \sum_{i=0}^{2}\sum_{j=0}^{2} b_{i,j}^2} \quad (23)$$

where

$$e_a = \sum_{i=0}^{2}(a_i - \widehat{a}_i)^2, \quad e_b = \sum_{i=0}^{2}\sum_{j=0}^{2}(b_{i,j} - \widehat{b}_{i,j})^2$$

Figure 4 shows the convergence of $MSE_2(t)$ for Volterra ECLMS algorithm.

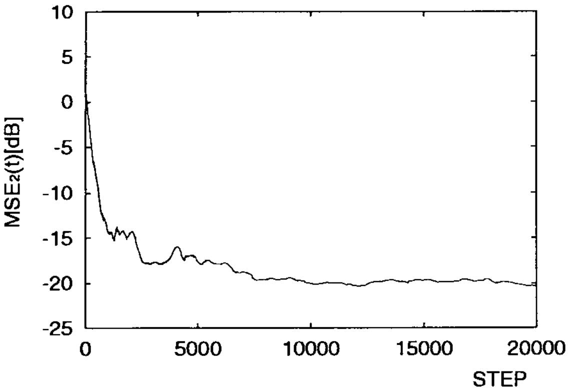

Fig.4. $MSE_2(t)$ of estimated filter coefficients.

The Volterra ECLMS algorithm shows steady convergence at about -20dB. Then we note that the Volterra ECLMS algorithm has good performance of coefficient estimation.

Next, we consider maximum likelihood (ML) estimation of the ARCH model. The log likelihood funcion[1][7] is as follows:

$$l = \frac{1}{T}\sum_{t=0}^{T}[-\frac{1}{2}\log(h(t)) - \frac{1}{2}\frac{\widehat{u}^2(t)}{h(t)}] \tag{24}$$

where

$$\widehat{u}(t) = y(t) - \widehat{\mathbf{A}}^T\mathbf{X}(t) - \mathrm{tr}[\widehat{\mathbf{B}}\mathbf{X}(t)\mathbf{X}^T(t)]$$

$$h(t) = \delta_0 + \delta_1\widehat{u}^2(t-1) + \delta_2\widehat{u}^2(t-2)$$

To measure the performance of the algorithm, we use $MSE(t)$ defined by the following equation:

$$MSE = 10\log_{10}\frac{e_a+e_b+e_\delta}{\sum_{i=0}^{2}a_i^2+\sum_{i=0}^{2}\sum_{j=0}^{2}b_{i,j}^2+\sum_{i=0}^{2}\delta_i^2} \tag{25}$$

where

$$e_a = \sum_{i=0}^{2}(a_i-\widehat{a}_i)^2, \quad e_b = \sum_{i=0}^{2}\sum_{j=0}^{2}(b_{i,j}-\widehat{b}_{i,j})^2$$

$$e_\delta = \sum_{i=0}^{2}(\delta_i-\widehat{\delta}_i)^2$$

The MSE of coefficients is shown in Table.1.

The proposed method(Volterra ECLMS + ML estimation) has a better convergence characteristics comparing with the ML estimation.

Table.1. MSE of coefficients.

	MSE
ML estimation($a_i, b_{i,j}, \delta_i$)	-13.25[dB]
VolterraECLMS($a_i, b_{i,j}$)+ML(δ_i)	-20.47[dB]

IV. Conclusion

We have proposed the estimation method for second-order Volterra model with ARCH errors. Coefficients in second-order Volterra model are calculated by Volterra ECLMS algorithm. Coefficients in ARCH process are calculated by ML estimation. The validity of the proposed method was verified by computer simulations.

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