On the Reachability Set of Petri Net under the Earliest Firing Rule

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Abstract: This paper studies coverability tree and reachability set of Petri net under the earliest firing rule. Conventional algorithm for coverability tree for 'normal' Petri net is not good for Petri net under the earliest firing rule. Moreover, it is shown that there exists no coverability graph for general class of earliest firing Petri net. Some subclasses are studied where coverability graph can be constructed.

1. Introduction

Petri net is one of effective models of discrete event systems. Coverability tree is one of analysis tools of Petri net[1]. On the other hand, introducing the earliest firing rule gives an interesting extension to the Petri net[2]. The rule forces enabled events to occur as soon as possible. Theoretically, it makes Petri net Turing machine equivalent. Practically, this rule gives a sub-optimal solution to scheduling problem.

In this paper coverability tree and reachability set of Petri net under the earliest firing rule are studied. Basic definitions and notations are presented in the section 2. An algorithm for coverability tree of Petri nets under the normal firing rule is reviewed in the section 3. This algorithm is not good for Petri net under the earliest firing rule. Moreover, it is shown that there exists no coverability graph for general class of earliest firing Petri net. In the section 4, a condition is derived that coverability graph can be constructed. The argument is based on periodicity behavior of the net. Some studies have been done on periodicity of earliest firing bounded marked graphs[3, 4, 5] and normal firing unbounded Petri nets[6]. This paper gives some extension to them: earliest firing unbounded conflict free net is studied.

Lastly, semi-linearity of the reachability set of earliest firing Petri net is studied in the section 5. It has been pointed out that if the reachability set of a Petri net is semi-linear, it can be efficiently calculated. While reachability set of conflict free net is known to be semi-linear under the normal firing rule, there exists an earliest firing conflict free net whose reachability set is not semi-linear.

2. Definition and Notations

Petri net is a 4-tuple (P, T, F, m_0). P and T are finite sets of nodes called places and transitions, respectively, $F \subseteq (P\times T)\cup(T\times P)$ is a set of arcs, and m_0 is the initial marking. Marking is a mapping from the set of places to nonnegative integers $\mathbf{N}$. A place p is said to have $m(p)$ tokens at the marking m. The pre-set and post-set of a node $x \in P\cup T$ are defined as ${}^\bullet x = \{y|(y,x) \in F\}$ and $x^\bullet = \{y|(x,y) \in F\}$.

A transition t is enabled at the marking m if all places of ${}^\bullet t$ have at least one token. This is denoted as $m[t\rangle$. Enabled transition may or may not fire. If a transition t fires at the marking m, it removes one token from each place of ${}^\bullet t$ and adds one token to each place of $t^\bullet$. This is denoted as $m[t\rangle m'$ if the resulting marking is m'. A marking m' is said to be reachable from the marking m if there exists a sequence of transitions $w = t_{j_1}t_{j_2}\cdots t_{j_q}$ such that $m[t_{j_1}\rangle m_1$, $m_1[t_{j_2}\rangle m_2$, ..., $m_{q-1}[t_{j_q}\rangle m'$. This is denoted as $m[w\rangle m'$. The reachability set $R(m_0)$ is the set of markings reachable from m_0.

A Petri net is a marked graph if $\forall p \in P$; $|{}^\bullet p| = |p^\bullet| = 1$. A Petri net is a conflict free net if $\forall p \in P$; $|p^\bullet| \leq 1$. A Petri net is an asymmetric choice net if $\forall p_1, p_2 \in P$; $p_1^\bullet \cap p_2^\bullet \neq \emptyset \rightarrow (p_1^\bullet \subseteq p_2^\bullet \vee p_1^\bullet \supseteq p_2^\bullet)$. An earliest firing net is a free firing net if no conflict occurs in every reachable marking. Although every earliest firing conflict free net is a free firing net, every free firing net is not a conflict free net.

A set of n-dimensional nonnegative integer vector L is linear if it can be expressed as

$$L = \{v_0 + k_1v_1 + \cdots + k_sv_s | k_j \in \mathbf{N}, v_j \in \mathbf{N}^n\}. \quad (1)$$

A set is semi-linear if it can be expressed as a union of finite number of linear sets.

The earliest firing rule forces enabled transition to fire. This rule implies discrete time on which transitions can fire. This also implies that more than one transition could fire simultaneously. The marking of the earliest firing Petri net changes with a firing of set or multiset of transitions called a step. If each transition can fire at least once at a time (single server), then the steps are sets of transitions. On the other hand if each transition can fire more than once at a time (infinite server), then steps are multisets of transitions. A single server transition can be simulated by a infinite server transition. It is done by augmenting infinite server transition t with one place t_p with one token and two arcs of (t, p_t) and (p_t, t). So transitions are assumed to be infinite servers unless otherwise stated.

The reachability set under the earliest firing rule is denoted as $R_e(m_0)$. Given a firing sequence of an earliest firing Petri net, let $S_w(k,\ t)$ be the number of firing of the transition t during the period $[0,\ k]$. $S_w(k)$ is $|T|$-dimensional vector whose elements are $S_w(k,\ t)$. A firing sequence is periodic if there exist a time instance k_0, period k_p and $|T|$-dimensional vector Δ such that

$$S_w(k + k_p) = S_w(k) + \Delta; \quad \forall k \geq k_0. \quad (2)$$

3. Coverability Graph

A coverability tree is one of the analysis tools of Petri net. It is a labeled finite tree $\mathcal{CT} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$. $\mathcal{V}$ is a set of $|P|$-dimensional vector over $\mathbf{N} \cup \{\omega\}$, where the symbol ω implies infinity: for each integer $n \in \mathbf{N}$

$$\omega > n, \ \omega \pm n = \omega, \ \omega \geq \omega \tag{3}$$

hold. $\mathcal{E}$ is a set of edges and $\mathcal{L}$ is labeling function $\mathcal{E} \mapsto T$. The root of the tree is m_0. $m \in R(m_0)$ implies $m' \in \mathcal{V}$ with $m'(p) = m(p)$ or $m'(p) = \omega$. $m' \in \mathcal{V}$ implies $\forall n \in \mathbf{N}$; $\exists m \in R(m_0)$ with $m(p) = m'(p)$ $(m'(p) < \omega)$ and $m(p) \geq n$ $(m'(p) = \omega)$. A path of the tree corresponds to a firing sequence. The procedure Coverability_tree in List 1 yields a coverability tree of a Petri net. In the algorithm, $\mathcal{M}$ denotes the set of internal nodes and $\mathcal{A}(m)$ denotes the set of ancestor nodes of m. A coverability graph is obtained by merging identical nodes of a coverability tree into one node.

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1:  V ← {m0}
2:  M ← V
3:  while M is nonempty do
4:    foreach m ∈ M do
5:      foreach t enabled at m do
6:        let m' be the marking such that m[t⟩m'
7:        if ∃m'' ∈ A(m); m'' ≤ m' then do
8:          foreach p such that m''(p) < m'(p) do
9:            m'(p) ← ω
10:         end
11:       end
12:       if (m' ∉ V) then M ← M ∪ {m'}
13:       V ← V ∪ {m'}
14:       E ← E ∪ {(m, m')}
15:       L(m, m') ← t
16:     end
17:   end
18: end
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List 1: Procedure Coverability_tree

The marking can be replaced by ω since $m''[w\rangle$ and $m' \geq m''$ imply $m'[w\rangle$, and the procedure terminates thanks to this substitution.

However, care must be taken under the earliest firing rule. Since fireable transition must fire under the earliest firing rule, $m''[w\rangle$ and $m' \geq m''$ does not always imply $m'[w\rangle$, Thus the line 7 of the procedure Coverability_tree should be modified as follows. We will call the modified procedure Earliest_coverability_tree. Let $w_{m \to m'}$ be the sequence from m to m' on the coverability tree.

$7'$: if $(\exists m'' \in \mathcal{A}(m); m'' \leq m')$ and $(m'[w_{m'' \to m'}\rangle)$ and $(m'[w\rangle \Leftrightarrow \forall n \in \mathbf{N}; (m' + n(m' - m''))[w\rangle)$ then do

The condition of $7'$ is very hard to verify. The following is a more restricting but easy-to-verify condition.

$7''$: if $(\exists m'' \in \mathcal{A}(m); m'' \leq m')$ and $(w_{m'' \to m'}$ is the unique prefix of firing sequences from $m')$ then do

Proposition 1 Condition in $7'$ holds if condition in $7''$ holds.

(Brief Proof) If the condition in $7''$ holds, then only legal firing from m' is $w_{m'' \to m'} w_{m'' \to m'} \cdots$. □

For example, consider the Petri net of the figure 1 with the initial marking $m_0 = (0\ 0\ 0\ 1\ 0)$. Figure 2 shows the coverability graphs of the net under the normal (a) and the earliest firing rule (b) respectively. Under both firing rule, $w_1 = t_1 t_3$ is a legal firing sequence from m_0 and the resulting marking $m = (0\ 1\ 0\ 1\ 1)$ satisfies $m \geq m_0$. Since $m[w_1\rangle$ is not legal under the earliest firing rule, we cannot substitute $m(p_2)$ nor $m(p_5)$ with ω. Indeed, the following shows that p_5 is bounded under the earliest firing rule. After the firing of a legal firing sequence $w_2 = \{t_1, t_4\}\{t_2, t_3\}$, a marking $m' = (0\ 2\ 0\ 1\ 1)$ is reached and $m' \geq m$ holds. Since $m'[w_2\rangle$ holds and $\{t_1, t_4\}\{t_2, t_3\}\{t_1, t_4\}\{t_2, t_3\} \cdots$ is only feasible sequence from m', we can substitute $m'(p_2)$ with ω.

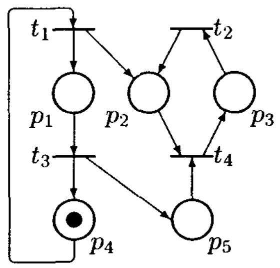
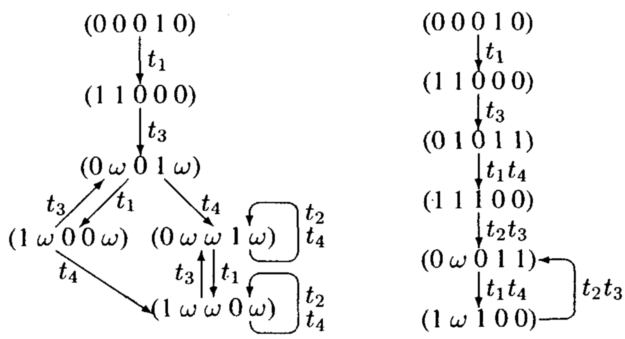


Figure 1: A Petri net



(a) Normal firing rule (b) Earliest firing rule

Figure 2: Coverability graphs

4. Coverability Graphs of Earliest Firing Petri Nets

Unfortunately, this procedure Earliest_coverability_tree does not always terminate.

Proposition 2 Earliest firing asymmetric choice net can simulate register machine regardless its transitions are single servers or infinite servers. [7].

Proposition 3 Free firing net can simulate register machine regardless its transitions are single servers or infinite servers. [8].

Since terminating problem of Turing machine can be reduced to coverability problem of these subclasses of the earliest firing Petri net, the procedure Earliest_coverability_tree is not valid for them.

The procedure is valid for simpler classes. We can show that conflict free net exhibit periodic behavior if all transitions can fire at most finite times at once. Then coverability tree can be made by using the procedure Earliest_coverability_tree with line 7''. Give a conflict free net N, the following procedure finds transitions that can fire infinitely many times at once. Without loss of generality, N can be assumed to be live.

1: Remove the places having two or more input arcs. Let the resulting marked graph be N'.
2: Decompose N' into strongly connected components. For every strongly connected component $N'_j = (P_j, T_j, F_j, m_{0j})$, calculate f_j by

$$f_j = \min_{y \in Y_j} \frac{y m_{0j}}{\sum_{p \in P_j} y(p)}, \qquad (4)$$

where Y_j is the set of S-invariants of N'_j. If $P_j = \emptyset$, let $f_j = \infty$.
3: Let $x_0(t) = f_j$ for $t \in T_j$.
4: Let T_{f0} be the set of transitions with finite value of $x(t)$. Let $T_f = T_{f0}$ and $P_f = \{p | {}^\bullet p \subseteq T_f\}$.
5: Update T_f and P_f as following until no change occurs in T_f.

$$T_f \leftarrow T_f \cup P_f^\bullet \qquad (5)$$
$$P_f \leftarrow \{p | {}^\bullet p \subseteq T_f\}. \qquad (6)$$

6: Let $T_\omega = T - T_f$.
7: Solve the following linear programming problem LP with variables $\{x(t) | t \in T_f\}$.

$$\max \sum_{t \in T_f} x(t)$$
$$x(t) \le \min_{p \in {}^\bullet t} \sum_{t' \in {}^\bullet p} x(t') \qquad (7)$$
$$x(t) \le x_0(t) \qquad (8)$$

The solution x^* of LP gives the upper bound of firing frequencies of the transitions in T_f. First, x^* is shown to be bounded.

Lemma 1 The linear programming problem LP has a unique optimal solution x^*.

(Proof) x^* is bounded. Indeed if $t \in T_{f0}$, t is bounded because of the inequality (8). If $t \in T_f - T_{f0}$, induction of the number of applying the updating procedure (5) shows that t is bounded thanks to the inequality (7). Now assume that there exist two distinct optimal solutions x_1 and x_2. Then x_3 defined by $x_3(t) = \max\{x_1(t), x_2(t)\}$ is a feasible solution having greater objective function than both x_1 and x_2. This makes a contradiction. □

Now boundedness of simultaneous firing number of $t \in T_f$ is derived.

Lemma 2 Let N be a live conflict free net and t be a transition in T_f. t can fire at most finite times simultaneously.

(Proof) Let x be the vector of average firing counts of the transitions. The inequalities (7) (8) of the linear programming problem must hold because of the balance of token count of the places. □

The converse of the lemma 2 is also true.

Lemma 3 Let N be a live conflict free net and t be a transition in T_ω. For any integer n, there exists a marking $m \in R_e(m_0)$ where t can fire more than n times simultaneously.

(Proof) Every cycle containing only nodes in $T_\omega \cup {}^\bullet T_\omega$ has at least one place p with $|{}^\bullet p| \ge 2$. Token count of such cycle is increasing since N is live. If t can fire at most finite times simultaneously, then t has a bounded input place p. Every transition of ${}^\bullet p \cap T_\omega$ can fire at most finite times simultaneously. Repeating this argument, a cycle is found whose places are bounded. This makes a contradiction. □

Now we prove the main theorem.

Theorem 1 Let N be a live conflict free net. N exhibits periodic behavior if and only if T_ω is empty,

(Proof) First we prove if part. Since N is conflict free, the firing sequence of N is unique. Let m_k be the marking at time instance k. Since N is live, there exists a nondecreasing subsequence of marking

$$m_{k_1} \le m_{k_2} \le m_{k_3} \le \cdots,$$

with $k_1 < k_2 < k_3 < \cdots$. Let δ be an arbitrary nonnegative integer δ and $F(k, \delta) = S(k + \delta) - S(k)$. Then $F(k_i, \delta) \le F(k_j, \delta)$ holds for any $i < j$. Since T_ω is empty, $F(k, \delta)$ is bounded (see lemma 2). Therefore, $F(k, \delta)$ converges. This means that there exist a suffix j and a sequence v such that

$$m_{k_j}[v\rangle m_{k_{j+1}} \wedge m_{k_{j+1}}[v\rangle.$$

Since N is conflict free, $m_{k_j}[v^n\rangle$ for any $n \ge 0$ and this is only legal firing sequence form m_{k_j}. Now only if part is proved. Assume that $T_\omega \ne \emptyset$ and N behaves periodically. However, this contradicts with the result of lemma 3. □

Corollary 1 Let N be a live conflict free net. If T_ω is empty, the procedure Earliest_coverability_tree terminates.

(Proof) N has unique firing sequence and it is periodic. Thus, the proof of the theorem 1 shows that there exist a suffix j, period c and a sequence v such that

$$m_{k_j}[v\rangle m_{k_j+c} \wedge m_{k_j+c}[v\rangle m_{k_j+2c}. \qquad (9)$$

Since the condition of the line 7'' holds, we can replace $m_{k_j+c}(p)$ with ω if $m_{k_j+c}(p) > m_{k_j}(p)$. It is clear from the equation (9) that $m_{k_j+2c}(p) = m_{k_j+c}(p)$ if $m_{k_j+c}(p) = m_{k_j}(p)$ and $m_{k_j+2c}(p) > m_{k_j+c}(p)$ if $m_{k_j+c}(p) > m_{k_j}(p)$. Thus m_{k_j+c} and m_{k_j+2c} are the identical nodes in the coverability tree and the procedure Earliest_coverability_tree terminates. □

Corollary 2 The procedure Earliest_coverability_tree terminates for the following subclasses of earliest firing Petri nets: (1) marked graphs having no transitions without input places and (2) single server conflict free nets.

(Proof) T_ω is empty for these subclasses of Petri net. □

5. Reachability Set of Earliest Firing Petri Net

If the reachability set $R_e(m_0)$ is semi-linear, it is likely that the algorithm Earliest_coverability_tree terminates. Even under the normal firing rule, the reachability set $R(m_0)$ is not semi-linear. It is known that the reachability set $R(m_0)$ of a conflict free net is semi-linear. However, there exists an earliest firing conflict free net of which reachability set $R_e(m_0)$ is not semi-linear if transitions are infinite server.

For example, the Petri net of the figure 3 is a conflict free net. The reachability set under the earliest firing rule is $R_e(m_0) = \{(2^n, 0, 0), (0, 2^n, 2^n) | n \geq 0\}$. This set is not semi-linear.

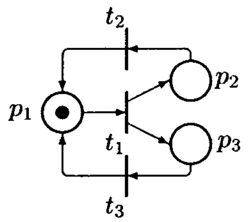


Figure 3: A conflict free net

If conflict free net behaves periodically, the reachability set is semi-linear.

Theorem 2 The reachability set of the earliest firing conflict free net N is semi-linear if T_ω is empty.

(Proof) Let m_k be the marking of N at the time instance k. If T_ω is empty, then the theorem 1 shows that N behaves periodically. So there exist time instance k_0, period k_p, and nonnegative vector δm such that

$$m_{(k+k_p)} = m_k + \delta m; \quad \forall k \geq k_0 \tag{10}$$

Then the reachability set is expressed as

$$R_e(m_0) = \bigcup_{k=0}^{k_0-1} \{m_k\} \cup \bigcup_{k=k_0}^{k_0+k_p-1} \{m_k + n \cdot \delta m | n \geq 0\} \tag{11}$$

which is a semi-linear set. □

Corollary 3 The reachability sets of the following subclasses of earliest firing Petri nets are semi-linear. (1) marked graphs having no transitions without input places and (2) single server conflict free nets.

(Proof) T_ω is empty for these subclasses of Petri net. □

6. Conclusion

This paper studies reachability set of earliest firing conflict free net. Condition to be able to construct a coverability graph is derived based on periodic behavior of the net. Future study includes finding whether the coverability tree can be constructed for general conflict free net under the earliest firing rule or not.

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