STATISTICAL ANALYSIS OF AN AUTOCALIBRATION METHOD FOR QUADRATURE RECEIVERS

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ABSTRACT

This paper deals with the statistical analysis of an autocalibration procedure for the gain and phase imbalances between the in-phase (I) and quadrature (Q) components in quadrature receivers. In real implementation, the imbalances of the gain and phase exist and degrade the performance of the receiver. In this paper we investigate the statistical characteristic of the estimates in an on-line imbalance estimation method for the receiver under the assumption of an additive white Gaussian noise environment.

1. INTRODUCTION

Quadrature receivers are widely used in communication and array signal processing. The exact match in the gain and phase of the I and Q channels of quadrature receivers is very important to achieve its maximum performance [1]. When the two parameters of the two channels are exactly matched, it usually becomes a classical parameter estimation problem in the complex random process which belongs to the Goodman class. However, practical quadrature receivers can not be ideal so that the receiver has gain and phase imbalances due to the mismatch. In this case, the classical signal processing algorithms may not work well.

The error analysis in a coherent detector was done by Sinsky and Wang [2]. Also, a correction method and its performance analysis for a coherent receiver under a noisy circumstance, were introduced by Churchill *et al* [3]. However, in their work the parameters of the testing input signal are assumed to be known. In this case the on-line calibration of the receiver is impossible.

The autocalibration algorithm proposed in [1] is useful even for the case that the parameters of the input signals are unknown, since the relative differences in the magnitude and phase are only used in their calibration algorithm.

This paper is organized as follows. In Section 2 the receiver model is introduced. The distribution, mean, and variance functions of the sampled data outputs in the imbalance estimation procedure are derived in Section 3 and Section 4. Then a brief discussion follows in the last section.

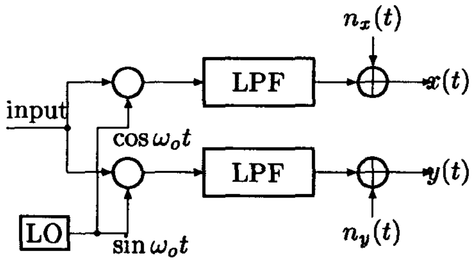

Figure 1: Quadrature receiver model.

2. RECEIVER MODEL

A quadrature receiver is shown in Figure 1. For a sinusoidal input signal of known carrier frequency ω_c(rad/sec) and unknown amplitude and phase, the outputs of the quadrature receiver may be described as

$$x(t) = A\cos(\omega_1 t + \psi) + n_x(t) \quad (1)$$
$$y(t) = A(1+\epsilon)\sin(\omega_1 t + \psi + \phi) + n_y(t) \quad (2)$$

where ϵ and ϕ are a relative fractional gain imbalance and a relative phase imbalance, respectively, and $\omega_1 = \omega_c - \omega_o$. Additive white Gaussian noises $n_x(t)$ and $n_y(t)$ are assumed to have the covariance of σ^2.

The imbalance estimation procedure for the autocalibration can be represented as in Figure 2. The outputs, $\hat{\epsilon}$ and $\hat{\phi}$, denote the estimates of the relative gain and phase imbalances, respectively.

3. DISTRIBUTION OF THE SAMPLED DATA OUTPUT, X_K AND Y_K

Sampling the output signals of the receiver at a time $(k+1)T$ where T is an integer multiple of $\frac{2\pi}{\omega_1}$, we have

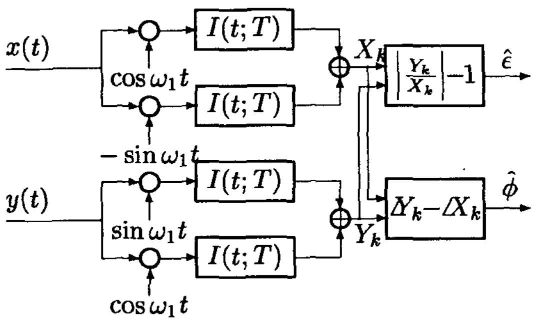

Figure 2: Imbalance estimation procedure where $I(t;T) = \frac{j}{T}\int_{kT}^{(k+1)T}(\cdot)\,dt$.

a complex Gaussian random variable pair, X_k and Y_k, given by

$$X_k = \frac{A}{2}e^{j\psi} + \frac{1}{T}\int_{kT}^{(k+1)T} n_x(t)e^{-j\omega_1 t}\,dt$$

$$Y_k = \frac{(1+\epsilon)A}{2}e^{j(\psi+\phi)} + \frac{1}{T}\int_{kT}^{(k+1)T} n_y(t)e^{+j\omega_1 t}\,dt.$$

Therefore, the distributions of X_k and Y_k are

$$X_k \sim CN\left(\frac{A}{2}e^{j\psi}, \frac{\sigma^2}{T}\right) \quad (3)$$

$$Y_k \sim CN\left(\frac{(1+\epsilon)A}{2}e^{j(\psi+\phi)}, \frac{\sigma^2}{T}\right). \quad (4)$$

If we get N sample pairs and denote them as $\mathbf{X} = [X_0, X_1, \cdots, X_{N-1}]$ and $\mathbf{Y} = [Y_0, Y_1, \cdots, Y_{N-1}]$, then the probability density functions of $\mathbf{X}$ and $\mathbf{Y}$ are given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{T^N}{\pi^N\sigma^{2N}}\exp\left\{-\frac{T}{\sigma^2}\sum_{k=0}^{N-1}|x(k) - m_{x(k)}|^2\right\}$$

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{T^N}{\pi^N\sigma^{2N}}\exp\left\{-\frac{T}{\sigma^2}\sum_{k=0}^{N-1}|y(k) - m_{y(k)}|^2\right\}$$

where

$$m_{x(k)} = \frac{A}{2}e^{j\psi}, \quad m_{y(k)} = \frac{(1+\epsilon)A}{2}e^{j(\psi+\phi)}, \quad (5)$$

for $k = 0, 1, \cdots, N-1$. Now, we define the sampled data using the polar coordinate as follows.

$$\mathbf{X} = \mathbf{x}_r + j\mathbf{x}_i = \mathbf{r}_{\mathbf{x}}e^{j\Theta_{\mathbf{x}}} \quad (6)$$

$$\mathbf{Y} = \mathbf{y}_r + j\mathbf{y}_i = \mathbf{r}_{\mathbf{y}}e^{j\Theta_{\mathbf{y}}}. \quad (7)$$

Then, from the joint densities of the each new random variable pair, we can have the marginal densities of each random vectors given by

$$f(\mathbf{r}_{\mathbf{x}}) = \frac{2^N T^N \prod_{k=0}^{N-1} r_{x_k}}{\sigma^{2N}} \cdot \exp\left\{-\frac{T}{\sigma^2}\sum_{k=0}^{N-1}(r_{x_k}^2 + A^2/4)\right\}\prod_{k=0}^{N-1} I_o\left(\frac{TAr_{x_k}}{\sigma^2}\right), \quad (8)$$

$$f(\Theta_{\mathbf{x}}) = \left(\frac{1}{2\pi}\right)^N \exp\left\{-\frac{A^2T}{4\sigma^2}\sum_{k=0}^{N-1}\sin^2(\theta_{x_k} - \psi)\right\} \cdot \prod_{k=0}^{N-1}\left[e^{-\frac{b_k^2}{2}} + \sqrt{2\pi}b_k\Phi(b_k)\right], \quad (9)$$

and

$$f(\mathbf{r}_{\mathbf{y}}) = \exp\left\{-\frac{T}{\sigma^2}\sum_{k=0}^{N-1}(r_{y_k}^2 + (1+\epsilon)^2A^2/4)\right\} \cdot \frac{2^N T^N}{\sigma^{2N}}\prod_{k=0}^{N-1} r_{y_k}\prod_{k=0}^{N-1} I_o\left(\frac{T(1+\epsilon)Ar_{y_k}}{\sigma^2}\right), \quad (10)$$

$$f(\Theta_{\mathbf{y}}) = \exp\left\{-\frac{(1+\epsilon)^2A^2T}{4\sigma^2}\sum_{k=0}^{N-1}\sin^2(\theta_{y_k} - \psi - \phi)\right\} \cdot \left(\frac{1}{2\pi}\right)^N\prod_{k=0}^{N-1}\left[e^{-\frac{c_k^2}{2}} + \sqrt{2\pi}c_k\Phi(c_k)\right] \quad (11)$$

where $b_k = \frac{\sqrt{2T}A}{2\sigma}\cos(\theta_{x_k} - \psi)$, $c = \frac{\sqrt{2T}(1+\epsilon)A}{2\sigma}\cos(\theta_{y_k} - \psi - \phi)$, and $\Phi(v)$ is a cumulative distribution function for $N(0, 1)$.

4. STATISTICS OF THE ESTIMATES: ONE SAMPLE CASE

We denoted a pair of samples at time k as r_{x_k} and r_{y_k} in the previous section. Hereafter, we let $|X| = r_{x_k}$ and $|Y| = r_{y_k}$ just for the sake of convenience in notation.

Letting $V = |Y|/|X|$ and $\eta = (arg(Y) - arg(X))_{2\pi} = (\theta_y - \theta_x)_{2\pi}$, V and η then become the values of $1 + \hat{\epsilon}$ and $\hat{\phi}$, respectively. Now, note that, in the previous section,

$$|X| \sim \chi\left(\frac{A}{2}, \frac{\sigma^2}{2T}\right), \quad |Y| \sim \chi\left(\frac{(1+\epsilon)A}{2}, \frac{\sigma^2}{2T}\right). \quad (12)$$

Therefore, we have a density function for V given as [4]

$$f_V(v) = \frac{2\exp\left\{-\frac{A^2T}{4\sigma^2}\left(1 + (1+\epsilon)^2\right)\right\}v}{(v^2+1)^2} \cdot \sum_{j=0}^{\infty}\left(\frac{A^2T}{4\sigma^2(1+v^2)}\right)^j (1+j)!\sum_{i=0}^{j}\left(\frac{(1+\epsilon)^i v^i}{i!(j-i)!}\right)^2. \quad (13)$$

Thus we easily have the probability density function of the gain imbalance W from the relation, $W = V - 1$. This density function is shown in Figure 3.

We now consider the mean and variance of W.

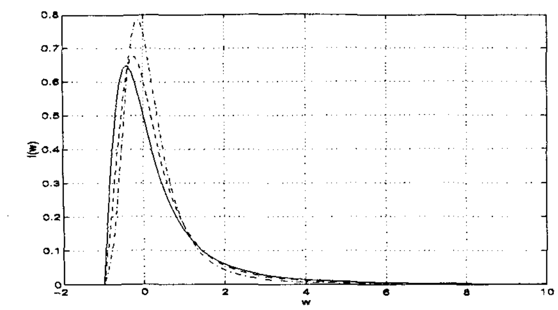

Figure 3: Density function of W when $T = 10^{-5}$sec (solid line), T = 5sec (dashed line), and T = 10sec (dash-dotted line). $A = 1$, $\epsilon = 0.1$, and $\sigma^2 = 1$ are assumed.

The r-th moment of $|Y|$ given by

$$E\{|Y|^r\} = \left(\frac{\sigma^2}{T}\right)^{r/2} \Gamma\left(\frac{r}{2}+1\right) \cdot {}_1F_1\left(-\frac{r}{2}; 1; -\frac{(1+\epsilon)^2A^2T}{4\sigma^2}\right) \quad (14)$$

where ${}_1F_1(\alpha; \beta; \pm u)$ is the confluent hypergeometric function [5].

Thus the first moment is given by

$$E\{|Y|\} = \frac{\sigma\sqrt{\pi}}{2\sqrt{T}} e^{-\frac{(1+\epsilon)^2A^2T}{8\sigma^2}} \left(1 + \frac{(1+\epsilon)^2A^2T}{4\sigma^2}\right) \cdot I_o\left(\frac{(1+\epsilon)^2A^2T}{8\sigma^2}\right) + \frac{\sigma\sqrt{\pi}}{2\sqrt{T}} e^{-\frac{(1+\epsilon)^2A^2T}{8\sigma^2}} \frac{(1+\epsilon)^2A^2T}{4\sigma^2} I_1\left(\frac{(1+\epsilon)^2A^2T}{8\sigma^2}\right). \quad (15)$$

Also [6], we have

$$E\left\{\left|\frac{1}{X}\right|\right\} = \frac{\sqrt{\pi T}}{\sigma} e^{-\frac{A^2T}{8\sigma^2}} I_o\left(\frac{A^2T}{8\sigma^2}\right). \quad (16)$$

Therefore, we obtain

$$E\{W\} = G(\epsilon, T)\left(1 + \frac{(1+\epsilon)^2A^2T}{4\sigma^2}\right) I_o\left(\frac{(1+\epsilon)^2A^2T}{8\sigma^2}\right) + G(\epsilon, T)\frac{(1+\epsilon)^2A^2T}{4\sigma^2} I_1\left(\frac{(1+\epsilon)^2A^2T}{8\sigma^2}\right) - 1, \quad (17)$$

where

$$G(\epsilon, T) = \frac{\pi}{2} e^{-\frac{A^2T}{8\sigma^2}(1+(1+\epsilon)^2)} I_o\left(\frac{A^2T}{8\sigma^2}\right). \quad (18)$$

Figure 4 shows $E\{W\}$ with respect to T and SNR. Note that the mean rapidly approaches the true value as T gets large. Thus we may conclude that this estimate is asymptotically unbiased. Actually this characteristic can be easily shown analytically since each of the first and second terms of (17) can be simplified for large T as

$$\text{first term} \simeq \frac{8\sigma^2}{4(1+\epsilon)A^2T} + \frac{1+\epsilon}{2} \quad (19)$$

$$\text{second term} \simeq \frac{1+\epsilon}{2} \quad (20)$$

using asymptotic expansion.

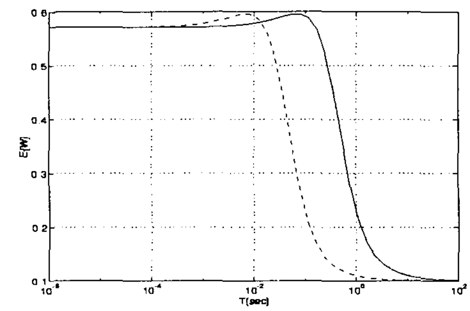

Figure 4: Mean of W as a function of T when SNR = $10dB$(solid line) and $SNR = 20dB$(dotted line). A = 1 and $\epsilon = 0.1$ are assumed.

The Variance of W can be derived similarly as

$$Var\{W\} = \left[\frac{\sigma^2}{T} + \frac{(1+\epsilon)^2A^2}{4}\right] \cdot \int_0^\infty \frac{2T}{\sigma^2 x} \exp\left\{-\frac{T}{\sigma^2}(x^2 + A^2/4)\right\} \cdot I_o\left(\frac{TAx}{\sigma^2}\right) dx - [E\{V\}]^2. \quad (21)$$

A numerical calculation of the variance of W, which is shown in Figure 5, shows the rapid approach of the variance to zero as T becomes large. Thus, it may be concluded that this estimate is also consistent.

The density function of η is

$$f_\eta(\eta) = \quad (22)$$

$$\int_0^{2\pi} \left[\frac{e^{-\frac{A^2T}{4\sigma^2}}}{2\pi} + \frac{A\sqrt{T}\cos\alpha}{2\sigma\sqrt{\pi}} e^{-\frac{A^2T}{4\sigma^2}\sin^2\alpha} \Phi\left(\frac{A\sqrt{T}}{\sigma\sqrt{2}}\cos\alpha\right)\right]$$

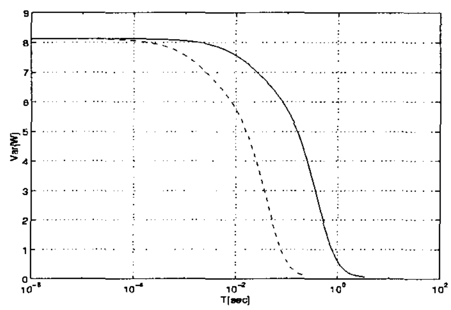

Figure 5: Variance of W as a function of T when SNR = $10dB$(solid line) and SNR = $20dB$(dotted line). $A = 1$ and $\epsilon = 0.1$ are assumed.

$$\cdot \left[\frac{e^{-\frac{A^2(1+\epsilon)^2 T}{4\sigma^2}}}{2\pi} + S(\beta;T)\Phi\left(\frac{A(1+\epsilon)\sqrt{T}}{\sigma\sqrt{2}}\cos\beta\right)\right] d\theta_x \tag{23}$$

where $\alpha = \theta_x - \psi$, $\beta = \eta + \theta_x - \psi - \phi$, and

$$S(\beta;T) = \frac{A(1+\epsilon)\sqrt{T}\cos\beta}{2\sigma\sqrt{\pi}} e^{-\frac{A^2(1+\epsilon)^2 T}{4\sigma^2}\sin^2\beta}. \tag{24}$$

A numerical example of this function may be shown as in Figure 6.

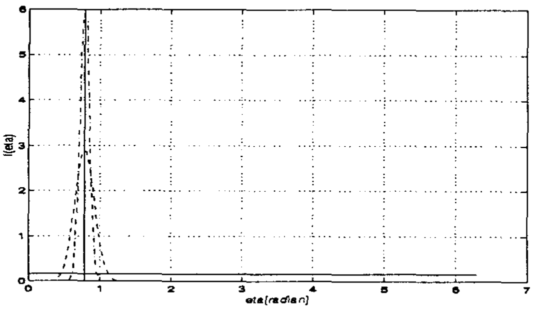

Figure 6: Probability density function of η when $T = 10^{-5}$sec (horizontal solid line), $T = 1$sec (dashed line), and $T = 5$sec (dash-dotted line). $\psi = 0, \phi = \pi/4$ (vertical solid line),$A = 1, SNR = 20dB$, and $\epsilon = 0.1$ are assumed.

Also, an example of the numerical calculations of the mean and variance of η is shown in Figure 7 and Figure 8, respectively. We can see that the mean and variance approach 0.1 and zero, respectively, as T gets large. This may imply that the estimate of the phase imbalance is consistent and asymptotically unbiased.

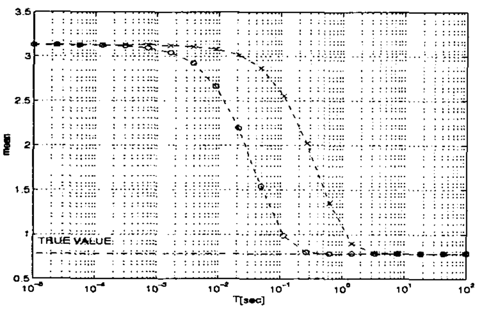

Figure 7: Mean of η as a function of T when $SNR = 10dB$ ('x'-marked line) and $SNR = 20dB$ ('o'-marked line). $\psi = 0, \phi = \pi/4, A = 1$, and $\epsilon = 0.1$ are assumed.

5. DISCUSSION

We analyzed the statistical characteristic of the estimates of the magnitude and phase imbalances in an autocalibration method that can be used in quadrature receivers. From the analytical and numerical results we may conclude that both are the minimum variance unbiased estimates asymptotically with respect to the variable T. A minimum sampling period T may be derived to guarantee a certain level of the required bit error probability for the receiver if the worst-case SNR is given. Also the recommended values of T corresponding to each SNR may be listed in a table to be used as a look-up reference in real applications.

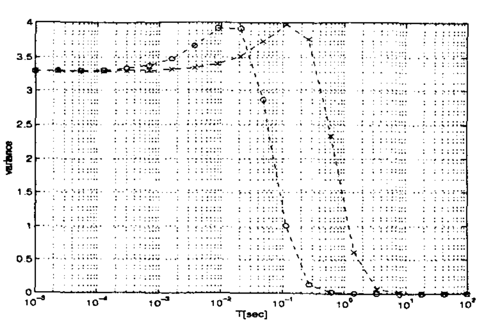

Figure 8: Variance of η as a function of T when $SNR = 10dB$ ('x'-marked line) and $SNR = 20dB$ ('o'-marked line). $\psi = 0, \phi = \pi/4, A = 1$, and $\epsilon = 0.1$ are assumed.

6. REFERENCES

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