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CUBIC B-SPLINE을 이용한 고유치 해석

EIGENVALUE ANALYSIS USING PIECEWISE CUBIC B-SPLINE

김 영 문*

Kim, Young-Moon

ABSTRACT

This paper presents properties of piecewise cubic B-spline function and Rayleigh-Ritz method to compute the smallest eigenvales. In order to compute the smallest eigenvalues, Rayleigh quotient approach is used and four different types of finite element approximating functions corresponding to the statical deflection curve, spanned by the linearly independent set of piecewise cubic B-spline functions with equally spaced 5 knots from a partion of [0, 1], each satisfying homogeneous boundary conditions with constraining effects are used to compute the smallest eigenvalues for a Sturm-Lionville boundary equations of $u'' + \lambda^2 u = 0,\ u(0.0) = u(1.0) = 0,\ 0 \le x \le 1.0$.

1. INTRODUCTION

In 1943, Courant suggested using piecewise linear functions to define the relevant subspaces of approximate trial functions. Since 1945, Schoenberg first introduced the idea of spline function, Garabedian and Birkhoff proposed using twice-differentiable piecewise cubic spline functions for the subspaces of approximate trial functions in 1960. Today numerical method using piecewise polynomials has become an active field of mathematics and engineering science. Piecewise cubic B-spline functions are class of piecewise polynomial functions which satisfy smoothness properties. They have useful properties for computing, which made them a powerful polynomial functions for numerical approximation, interpolation and numerical solution for differential equations etc.

In this paper, four different types of finite element approximating functions spanned by piecewise cubic B-spline functions, each satisfying homogeneous boundary conditions and representing contraining effect of boundary conditions are used· to compute the smallest eigenvalues for Sturm-Liouville· equation. Rayleigh-quotient approach is used to compute numerical values of the smallest eigenvalues.

* 정회원 · 전북대학교 건축 · 도시공학부 · 공업기술연구소 교수

2. PIECEWISE CUBIC B-SPLINE BASIS FUNCTION (C.B.S)

2-1. Properties of C.B.S

C.B.S $B_i(x)$ is a piecewise twice continuously differentiable and locally supported such as shown in Fig. 1. So, it is very convenient to compute using computer.

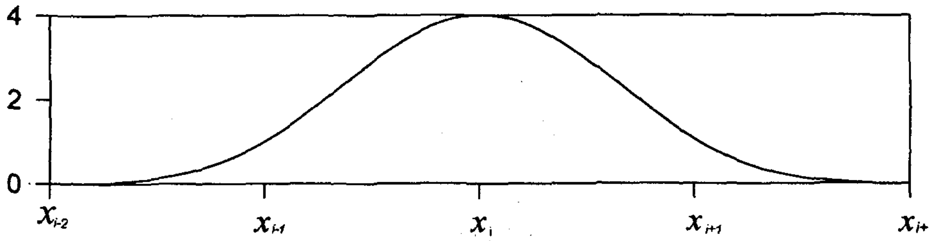

Fig. 1 Graph of C.B.S $B_i(x)$

The functions of each subinterval are such as below

$$B_i(x) = \begin{cases} \frac{1}{h^3}[(x-x_{i-2})^3] & \text{if } x \in [x_{i-2}, x_{i-1}] \\ \frac{1}{h^3}[h^3 + 3h^2(x-x_{i-1}) + 3h(x-x_{i-1})^2 - 3(x-x_{i-1})^3] & \text{if } x \in [x_{i-1}, x_i] \\ \frac{1}{h^3}[h^3 + 3h^2(x_{i+1}-x) + 3h(x_{i+1}-x)^2 - 3(x_{i+1}-x)^3] & \text{if } x \in [x_i, x_{i+1}] \\ \frac{1}{h^3}[(x_{i+2}-x)^3] & \text{if } x \in [x_{i+1}, x_{i+2}] \\ 0 & \textit{otherwise} \end{cases} \quad (1)$$

where $h = x_i - x_{i-1}$

And the table 1 presents the values of the $B_i(x)$ at 5 knots.

Table 1 Values of $B_i(x)$ at knots

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i'(x)$	0	3/h	0	−3/h	0
$B_i''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

2-2.Approximating Functions

In this paper, the finite element approximating functions U_N spanned by the lineally independent finite set of piecewise cubic B-spline functions $B_i(x)$ are assumed as ;

$$U_N = \sum_{j=1}^{N} C_j B_j(x) \qquad (2)$$

And it is possible to choose different types of U_N for satisfying given boundary condition. So, it may be rewritten as

$$U_N = \sum_{j=1}^{N} C_j \tilde{B}_j(x) \qquad (3)$$

In this paper, four different types of U_N spanned by the linearly independent set of piecewise cubic B-spline functions, with equally space knots from $0=x_1<x_2< \cdot\cdot\cdot <x_5=1.0$, each satisfying homogeneous boundary conditions of eq.(9) and representing constraining effect of boundary condition are assumed as:

Type 1 : $U_1 = \sum_{j=1}^{5} C_j \tilde{B}_j(x)$

where $\tilde{B}_1(x) = B_1(x) - 4B_0(x)$, $B_3(x) = B_3(x)$, $\tilde{B}_4(x) = B_5(x) - 4\tilde{B}_4(x)$, (4)
$\tilde{B}_2(x) = B_1(x) - 4B_2(x)$, $\tilde{B}_5(x) = B_5(x) - 4B_6(x)$,

Type 2 : $U_2 = \sum_{j=1}^{5} C_j \tilde{B}_j(x)$

where $\tilde{B}_1(x) = B_1(x) - 4B_0(x)$, $B_3(x) = B_3(x)$, $\tilde{B}_4(x) = \frac{1}{2} B_5(x) - 2\tilde{B}_4(x)$, (5)
$\tilde{B}_2(x) = \frac{1}{2} B_1(x) - 2B_2(x)$, $\tilde{B}_5(x) = B_5(x) - 4B_6(x)$,

Type 3 : $U_3 = \sum_{j=1}^{5} C_j \tilde{B}_j(x)$

where $\tilde{B}_1(x) = B_1(x) - 4B_0(x)$, $B_3(x) = B_3(x)$, $\tilde{B}_4(x) = \frac{1}{4} B_5(x) - \tilde{B}_4(x)$, (6)
$\tilde{B}_2(x) = \frac{1}{4} B_1(x) - B_2(x)$, $\tilde{B}_5(x) = B_5(x) - B_6(x)$,

Type 4 : $U_4 = \sum_{j=1}^{5} C_j \tilde{B}_j(x)$

where $\tilde{B}_1(x) = B_1(x) - 4B_0(x)$, $B_3(x) = B_3(x)$, $\tilde{B}_4(x) = \frac{1}{8} B_5(x) - \frac{1}{2}\tilde{B}_4(x)$, (7)
$\tilde{B}_2(x) = \frac{1}{8} B_1(x) - \frac{1}{2} B_2(x)$, $\tilde{B}_5(x) = B_5(x) - 4B_6(x)$,

3. EIGENVALUE ANALYSIS FOR STURM-LIOUVILLE EQUATIONS

The Rayleigh-Ritz(or Galerkin) method has long been applied to compute approximate eigenvalues and eigenfunctions for Sturm-Liouville equations and other boundary value problems.

We recall that a Sturm-Liouville equation having homogeneous boundary condition is defined by a differential equation of the form

$$u'' + \lambda^2 u = 0, \qquad 0 \leq x \leq 1.0 \tag{8}$$

Subject to homogeneous boundary conditions, such as

$$u(0) = u(1.0) = 0 \tag{9}$$

Then, eq.(8) has a sequence of distinct eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 \cdot \ \cdot, \tag{10}$$

and a corresponding sequence of eigenfunctions.
These eigenvalues are stationary values and the eigenfunctions are critical points for the Rayleigh quotient $R(u)$ of eq.(8)~eq.(9) :

$$R(u) = \frac{N(u)}{D(u)} \tag{11}$$

$$\text{where} \quad N(u) = \int_0^{1.0} (u')^2 dx \ , \quad D(u) = \int_0^{1.0} u^2 dx \tag{12}$$

If $\phi_1(x)$ is the eigenfunction corresponding to the smallest eigenvalues λ_1^2, then $\phi_1(x)$ satisfies eq.(8) for $\lambda = \lambda_1$. We know of course that $\phi_1(x) = \sin \pi x$ and corresponding λ_1^2 for the Rayleigh quotient is π^2. The mode shapes(or eigenfunction) $\phi_1(x)$ corresponding to the smallest eigenvalue is usually the statical deflection curve satisfying boundary condition. In this paper, in order to compute the smallest eigenvalue λ_1^2 numerically, instead of using single eigenfunction $\phi_1(x)$ for the Rayleigh quotient, four different types of finite element approximating functions of eq.(3) corresponding to the statical deflection curve spanned by finite set of piecewise cubic B-spline functions satisfying homogeneous boundary conditions are used. The undetermined coefficients C_i's of four different types of finite element approximating functions could be determined by solving second order beam-bending differential equation having homogeneous boundary condition of eq.(13) based on Galerkin finite element method.

Consider the linear simply supported beam-bending differential equation having homogeneous boundary condition

$$Lu(x) = f(x) \qquad x \in \Omega \tag{13}$$

$$u(0) = u(1.0) = 0.0 \tag{14}$$

where $L = -\frac{d^2}{dx^2}$, positive definite symmetric linear differential operator

$u(x)$ = the unknown solution

Ω = the domain which the differential equation applies

$f(x) = \frac{12P}{24EI}(-Lx+x^2)$

$L=1.0m$, $E=10^6 kg/m^2$, $I=\frac{2}{3}m^4$, $P=100Kg/m$

Galerkin finite element method in the energy inner product space consists in finding the approximate solution U_N based on the least squares fit.

Applying Galerkin finite element principle to Eq.(13) using Eq.(3), then following linear system can be obtained.

$$\sum_{j=1}^{N}\int_0^{1.0}(\tilde{B}_i'(x)\ \tilde{B}_j'(x))C_j\,dx=\int_0^{1.0}f(x)\tilde{B}_i(x)\,dx \quad 1\le i\le N \tag{15}$$

Combining Eq.(3) (13) and (14), the approximating solution U_N satisfying boundary conditions can be determined by solving the linear system of Eq.(15) in terms of C_j and then these values are substituted into the Eq.(3) again to form the approximating solutions.

Values of C_j's obtained solving linear system of eq.(15) are shown as in Table 2 and substituting these values into eq(4) ~ eq(7). constitute the four finite element approximating solutions of Fig. 2.

Table 2 Values of C_j

	C_1	C_2	C_3	C_4	C_5
Type1	0.264944×10^{-6}	-0.257710×10^{-6}	0.1438630×10^{-5}	-0.257710×10^{-6}	0.264944×10^{-6}
Type2	0.392669×10^{-6}	-0.129984×10^{-6}	0.1387548×10^{-5}	-0.129984×10^{-6}	0.392669×10^{-6}
Type3	0.456531×10^{-6}	-0.66121×10^{-7}	0.1362007×10^{-5}	-0.66121×10^{-7}	0.456531×10^{-6}
Type4	0.488462×10^{-6}	-0.34189×10^{-7}	0.1349237×10^{-5}	-0.34189×10^{-7}	0.488462×10^{-6}

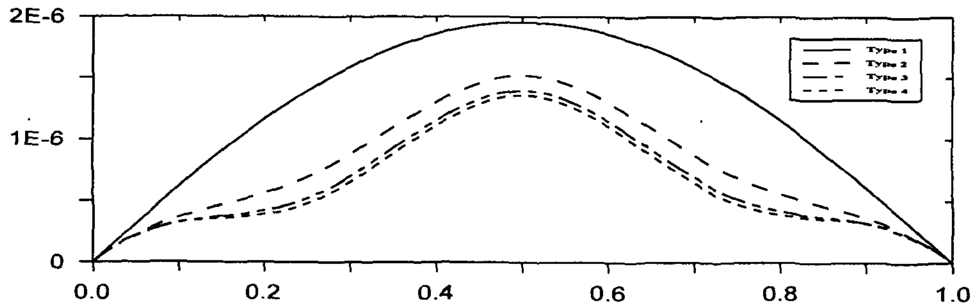

Fig.2.Solutions by Galerkin finite element method

4. FINITE ELEMET APPROXIMATING NUMERICAL RESULTS

Consider a Sturm-Liouville equation of eq.(8) having homogeneous boundary conditions of eq(9). In order to compute the smallest eigenvalue λ_1^2 by Rayleigh quotient of eq.(11), four different types of finite elemet approximating of functionw of eq.(4)~(7) with the values of C_j appeared in Table2 were used and four different values of λ_1^2's are shown in Table3. Numerical computation of eq.(11) was performed using the MATLAB.

Table3. Values of λ_1^2

Type	Analytic Value	Type 1	Type 2	Type 3	Type 4
λ_1^2	$\pi^2(9.8696)$	9.8696	12.9096	16.1540	17.3692

5. CONCLUSION

In this paper, four different types of finite element approximating functions spanned by piecewise cubic B-spline functions are applied to compute numerically the smallest eigenvalue. Comparing these approximated values with analytic value and obtained the following results. Values of the smallest eigenvalue calculated using piecewise cubic B-spline functions is good approximation to the analytic value of eigenvalue. And numerical algorithm for computing the smallest eigenvalues based on Rayleigh quotient for Sturm-Liouville equations having homogeneous boundary conditions using piecewise cubic B-spline functions is developed. And confirmed that to constraining effect of around the boundary conditions appears as increasing the value of eigenvalue.

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