퍼지 넘버 연산에 의한 퍼지 시계열 모형

Fuzzy time-series model of fuzzy number observations

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ABSTRACT

Recently, a homogeneous fuzzy time series model was proposed by means of defining some new operations on fuzzy numbers. In this paper, we consider expanding the results to the nonhomogeneous fuzzy time series and the general fuzzy time series using T_W, the weakest t-norm, based algebraic fuzzy operations.

1. Introduction

In 1993, a special dynamic process called fuzzy times series[5] is proposed due to the need for modelling dynamic processes whose observations are linguistic values[10]. The motivation of fuzzy set theory is maninly to provide a formal, powerful and quantitative framework to cope with the vagueness of human knowledge as it is expressed by means of natural languages[10]. In [6, 7], as an application of fuzzy time series, two models were used to forecast the entrollments of the University of Alabama by fuzzifying the historic data. Recently, Song et. al.[8] derived a new model in two forms for the homogeneous fuzzy time series by defining a new fuzzy addition and subtraction operations. And they mentioned that further studies are needed to expand the conclusitions to the nonnomegeneous fuzzy time series and the general fuzzy time series. In this paper, we reconsider the fuzzy addition and subtraction they defined in [8] and modify the fuzzy algebraic operations using extension principle[1]. And we expand the results in [8] to the general fuzzy time series.

2. Preliminaries

A function $T:[0,1]\times[0,1]\rightarrow[0,1]$ is said to be a t-norm [9] iff T is symmetric, associative, non-decreasing in each argument, and $T(x,1)=x$ for all $x\in[0,1]$. The usual arithmetical operation of reals can be extended to the arithmetical operations on fuzzy numbers by means of Zadeh's extension principle based on a triangular norm T. Let A, B be fuzzy numbers of the real line R. The fuzzy set arithmetic operations are summarized as follows:

Fuzzy set addition $\oplus$:

$$(A \oplus_T B)(z) = \sup_{x+y=z} T(A(x), B(y)).$$

Fuzzy set subtraction $\ominus$:

$$(A \ominus_T B)(z) = \sup_{x-y=x} T(A(x), B(x)).$$

If $T_1 \le T_2$(the usual order of t-norm as two-place function), then for any fuzzy quantity A and B it is $A \oplus_{T_1} B \le A \oplus_{T_2} B$. Let T_W denote the weakest t-norm defined by

$$T_W(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ 0 & \text{otherwise}, \end{cases}$$

and let T_M denote the strongest t-norm defined by $T_M(x, y) = \min(x, y)$ for all $x, y \in [0, 1]$. Consequently, for any t-norm T it is

$$A \oplus_W B \le A \oplus_T B \le A \oplus_M B$$

A fuzzy set A is called fuzzy interval if it is continuous and for each $\alpha \in (0, 1]$, the corresponding α-cut $A^\alpha = \{x \in R; A(x) \ge \alpha\}$ is a non-empty convex closed subset of R. If the support of A, $suppA = \bigcup A^\alpha$, is bounded in R, then the fuzzy interval A is so-called fuzzy interval. If, for an fuzzy interval A, $A^1 = [l_A, r_A]$ is a singleton (i.e., $l_A = r_A$), then A is called an fuzzy number.

Let $\Im$ be the class of all fuzzy quantities defined on $[0, \infty]$ which are continuous nonincreasing with the strict maximum 1 in the point 0 and its members be called shapes. Any fuzzy interval A can be written as a quadruple

$$A = (l_A, r_A, A_*, A^*)$$

where A_* and A^* are defined by $A^*(x) = A(x + r_A)$ and $A_*(x) = A(l_A - x)$, (if $l_A = -\infty$, then $A_* = \emptyset$; similarly if $r_A = \infty$ then $A^* = \emptyset$; otherwise $A_*, A^* \in \Im$),

$$A(x) = \begin{cases} 1 & \text{if } x \in [l_A, r_A], \\ A^*(x - r_A) & \text{if } x > r_A, \\ A_*(l_A - x) & \text{if } x < l_A. \end{cases}$$

It is known [4] that the addition and the opposite of fuzzy intervals is defined by

$$\begin{aligned} A \oplus_T B &= (l_A, r_A, A_*, A^*) \oplus_T (l_B, r_B, B_*, B^*) \\ &= (l_A + l_B, r_A + r_B, \\ &\qquad A_* \oplus_T B_*, A^* \oplus_T B^*), \end{aligned}$$

and $-A = (-r_A, -l_A, A^*, A_*)$

where, by the convention, $\infty + x = \infty$ and $-\infty + x = -\infty$ for all $x \in R$, and for any element S from $\Im \cup \{\emptyset\}$, $S \oplus_T \emptyset = \emptyset$.

Following that above argument, it suffices to study only about fuzzy arithmetic operations of shapes instead of fuzzy intervals.

Let N be a LR-fuzzy number whose membership function is defined as follows:

$$N(x) = \begin{cases} L((m - x)/a) & \text{for } x \le m, \ a > 0, \\ R((x - m)/b) & \text{for } x \ge m, \ b > 0, \end{cases}$$

where $L, R \in \Im$. Symbolically, we write

$$N = (m, a, b)_{LR}$$

The addition and subtraction of fuzzy sets based on strongest t-norm T_M is well known since the 1980s [1].

$$(m_1, a_1, b_1)_{LR} \oplus_M (m_2, a_2, b_2)_{LR}$$
$$= (m_1 + m_2, a_1 + a_2, b_1 + b_2)_{LR}$$

$$(m_1, a_1, b_1)_{LR} \ominus_M (m_2, a_2, b_2)_{RL}$$
$$= (m_1 - m_2, a_1 + b_2, b_1 + a_2)_{LR}$$

More generally, we have

Theorme 1 (Dubois and Prade [1]). Let A_1, A_2 be fuzzy intervals. Then for each $\alpha \in (0, 1]$,

$$(A_1 \oplus_M A_2)^\alpha = A_1^\alpha + A_2^\alpha.$$

For the weakest t-norm T_W, directly from the extension principle we have the following results:

Theorem 2 ([2, 3, 4]). Let $A_i=(l_i, r_i, L_i, R_i)$, $i=1,2$ be fuzzy intervals. Then

$$A_1 \oplus_W A_2 = (l_1+l_2, r_1+r_2, \max L_i, \max R_i),$$

and

$$A_1 \ominus_W A_2 = (l_1-r_2, r_1-l_2, \max\{L_1, R_2\}, \max\{R_1, L_2\}).$$

If A_1 and A_2 are LR-fuzzy numbers, then from Theorem 2, we have

$$(m_1, a_1, b_1)_{LR} \oplus_W (m_2, a_2, b_2)_{LR}$$

$$=(m_1+m_2, \max\{a_1, a_2\}, \max\{b_1, b_2\})_{LR}$$

and

$$(m_1, a_1, b_1)_{LR} \ominus_W (m_2, a_2, b_2)_{RL}$$

$$=(m_1-m_2, \max\{a_1, b_2\}, \max\{a_2, b_1\})_{LR}$$

In [8], Song et al. introduced new addition and subtraction operations for fuzzy numbers, denoted as $+_f$ and $-_f$, called linguistic addition and subtraction respectively, and defined below.

Definition [8]. For fuzzy numbers $A_1=(m_1, a_1, b_1)_{LR}$ and $A_2=(m_2, a_2, b_2)_{LR}$, their linguistic addition and subtraction are defined as follows:

$$A_1 +_f A_2 = (m_1+m_2, \max\{a_1, a_2\}, \max\{b_1, b_2\})_{LR}$$

$$A_1 -_f A_2 = (m_1-m_2, \max\{a_1, a_2\}, \max\{b_1, b_2\})_{LR}$$

Under this fuzzy arithmetic operations, Song et al. [8] derived a new model in two forms for the homogeneous fuzzy time series.

But, as we can see, the definition of linguistic addition and subtraction is slightly different from T_W-based fuzzy arithmetic operations. Song el al. [8] did not provide any reason why they defined as above. So, it is better to adopt T_W-based fuzzy arithmetic operations to study a model in two forms for the homogeneous fuzzy time series, aditionally for the general fuzzy series.

3. The models

Definition 1. Suppose $Y(t)$, $t\in T$, where $T=\{\cdots,0,1,2,\cdots\}$, is a subset of R and on it are defined some fuzzy sets $f_i(t)$, $(i=1,2,\cdots)$ and $F(t)$ is the collection of $f_i(t)$ $(i=1,2,\cdots)$. Then $F(t)$ is called a fuzzy time series on $Y(t)$, $t\in T$.

The following is a generalization of Proposition 1 of Song et al. [8].

Proposition. Let $\{N_i\}=\{(m_i, L_i, R_i)\}_{i\in I}$ be a sequence of fuzzy numbers where $I=\{1,2,\cdots\}$ with $\sum_{i=1}^{\infty} m_i < \infty$. Then $N=N_1 \oplus_W N_2 \oplus_W \cdots \oplus_W N_k \oplus_W + \cdots$ is a fuzzy number if and only if $\sup_i L_i^{\alpha} < \infty$ and $\sup_i R_i^{\alpha} < \infty$ for $0<\alpha<1$.

Definition 3. Let $F(t)$, $t\in T$, be a fuzzy time series with the observation $F(t)=(m(t), L(t), R(t))$ be fuzzy numbers. Define

$$D_b^1 F(t) = F(t) \ominus_W F(t-1)$$

where

$$D_b^1 F(t) = (m_{db}^1(t), L_{db}^1(t), R_{db}^1(t)),$$

$$m_{db}^1(t) = m(t) - m(t-1)$$

$$L_{db}^1(t) = \max\{L(t), R(t-1)\}$$

and

$$R_{db}^1(t) = \max\{R(t), L(t-1)\};$$

$$D_b^2 F(t) = D_b^1 F(t) \ominus_W D_b^1 F(t-1)$$

where

$$D_b^2F(t)=(m^2_{db}(t),L^2_{db}(t),R^2_{db}(t)),$$

$$m^2_{db}(t)=m^1_{db}(t)-m^1_{db}(t-1)$$

$$L^2_{db}(t)=\max\{L^1_{db}(t),R^1_{db}(t-1)\}$$

and

$$R^2_{db}(t)=\max\{R^1_{db}(t),L^1_{db}(t-1)\}.$$

In general,

$$D_b^kF(t)=D_b^{k-1}F(t)\ominus_W D_b^{k-1}F(t-1)$$

where

$$D_b^kF(t)=(m^k_{db}(t),L^k_{db}(t),R^k_{db}(t)),$$

$$m^k_{db}(t)=m^1_{db}(t)-m^1_{db}(t-1)$$

$$L^k_{db}(t)=\max\{L^{k-1}_{db}(t),R^{k-1}_{db}(t-1)\}$$

and

$$R^k_{db}(t)=\max\{R^{k-1}_{db}(t),L^{k-1}_{db}(t-1)\}.$$

Then $D_b^kF(t)$ is called the kth order ($k=1,2,\cdots$) backward liguistic difference of $F(t)$ at t.

Definition 4. Let $F(t)$, $t\in T$, be a fuzzy time series with the observation $F(t)=(m(t),L(t),R(t))$ be fuzzy numbers. Define

$$D^1F(t)=F(t+1)\ominus_W F(t)$$

where

$$D^1F(t)=(m^1_d(t),L^1_d(t),R^1_d(t)),$$

$$m^1_d(t)=m(t+1)-m(t)$$

$$L^1_d(t)=\max\{L(t+1),R(t)\}$$

and

$$R^1_d(t)=\max\{R(t+1),L(t)\};$$

$$D^2F(t)=D^1F(t+1)\ominus_W D^1F(t)$$

where

$$D^2F(t)=(m^2_d(t),L^2_d(t),R^2_d(t)),$$

$$m^2_d(t)=m^1_d(t+1)-m^1_d(t)$$

$$L^2_d(t)=\max\{L^1_d(t+1),R^1_d(t)\}$$

and

$$R^2_d(t)=\max\{R^1_d(t+1),L^1_d(t)\}.$$

In general,

$$D^kF(t)=D^{k-1}F(t+1)\ominus_W D^{k-1}F(t)$$

where

$$D^kF(t)=(m^k_d(t),L^k_d(t),R^k_d(t)),$$

$$m^k_d(t)=m^1_d(t+1)-m^1_d(t)$$

$$L^k_d(t)=\max\{L^{k-1}_d(t+1),R^{k-1}_d(t)\}$$

and

$$R^k_d(t)=\max\{R^{k-1}_d(t+1),L^{k-1}_d(t)\}.$$

Then $D^kF(t)$ is called the kth order ($k=1,2,\cdots$) backward liguistic difference of $F(t)$ at t. Both $D_b^kF(t)$ and $D^kF(t)$ are also a fuzzy time series for any $k>0$.

Definition 5. Let $F(t)$, $t\in T$, be a fuzzy time serises and $F(t)=(m(t),L(t),R(t))$. If $L(t)=L(t-1)$ and $R(t)=R(t-1)$ for any $t\in T$, then $F(t)$ is said to be a homogeneous fuzzy time series.

Lemma 1. Let $F(t)$ be a homoneous fuzzy time series, $t\in T$, and $F(t)=(m(t),L(t),R(t))$. Then

$$L(t)=L^k_{db}(t)=R^k_{ab}(t)=R(t)=L^k_d(t)=R^k_d(t),$$

$$t\in T,\ k=1,2,\cdots$$

For general fuzzy time series, we have the following first new model which indicates that the value of $F(t)$ at $t+1$ can be related to its previous ones by means of forward linguistic differences of various orders.

Theorem 1. Let $F(t)=(m(t),L(t),R(t))$, $t\in T$ be a fuzzy time series. Then we have

$$F(t+1)=F(t)\oplus_W D^1F(t-1)\oplus_W D^2F(t-2)\oplus_W \cdots\oplus_W D^k(t-k)\oplus_W\cdots$$

if and only if for any $t\in T$

$$m(t+1) = \sum_{i=0}^{\infty} m_d^i(t-i)$$
$$\text{where} \quad m_d^0(t-0) = m(t)$$

$$L(t+1) = \max\{\sup_{k \le t} L(k), \sup_{k \le t-1} R(k)\}$$

and

$$R(t+1) = \{\sup_{k \le t} R(k), \sup_{k \le t-1} L(k)\}.$$

Now, we consider the following lemma.

Lemma 2. For any $t \in T$, we have

$$\max\{\sup_{k \le t} L(k), \sup_{k \le t-1} R(k)\} = L(t+1)$$

and

$$\max\{\sup_{k \le t} R(k), \sup_{k \le t-1} L(k)\} = R(t+1)$$

if and only if for any $t \in T$

$$L(t-1) = L(t) = R(t) = R(t-1).$$

By Lemma 1 and Lemma 2, the following conclusion for the homogeneous fuzzy time series is immediate.

Theorem 2. Let $F(t)$, $t \in T$ be a fuzzy time series, then $F(t)$, $t \in T$ is homogeneous if and only if

$$F(t+1) = F(t) \oplus_W D^1 F(t-1) \oplus_W D^2 F(t-2) \oplus_W \cdots \oplus_W D^k(t-k) \oplus_W \cdots$$

$$\text{i.e.,} \quad F(t+1) = \left(\sum_{i=1}^{\infty} m_d^i(t-i), L(t+1), R(t+1)\right),$$

$$\text{and} \quad L(t-1) = L(t) = R(t) = R(t-1), \quad t \in T.$$

The following result is another form of the new model of general fuzzy time series which indicate that the value of $F(t)$ at $t+1$ can be expressed as the linguistic summation of various order linguistic backword difference at t.

Theorem 3. Let $F(t) = (m(t), L(t), R(t))$, $t \in T$ be a fuzzy time series. Then

$$F(t+1) = F(t) \oplus_W D_b^1 F(t) \oplus_W D_b^2 F(t) \oplus_W \cdots \oplus_W D_b^k(t) \oplus_W \cdots$$

if and only if for any $t \in T$

$$m(t+1) = \sum_{i=0}^{\infty} m^i_{ab}(t)$$
$$\text{where} \quad m^0_{ab}(t) = m(t)$$

$$L(t+1) = \max\{\sup_{k=1,2,\cdots} L(t-2k), \sup_{k=1,2,\cdots} R(t-1-2k)\}$$

and

$$R(t+1) = \max\{\sup_{k=1,2,\cdots} R(t-2k), \sup_{k=1,2,\cdots} L(t-1-2k)\}.$$

We also have the following result.

Theorem 4. Let $F(t)$, $t \in T$ be a fuzzy time series, then $F(t)$, $t \in T$ is homogeneous if and only if

$$F(t+1) = F(t) \oplus_W D_b^1 F(t) \oplus_W D_b^2 F(t) \oplus_W \cdots \oplus_W D_b^k(t) \oplus_W \cdots$$

$$\text{i.e.,} \quad F(t+1) = \left(\sum_{i=0}^{\infty} m^i_{ab}(t), L(t+1), R(t+1)\right),$$

$$\text{and} \quad L(t-1) = L(t) = R(t) = R(t-1), \quad t \in T.$$

4. Conclusion

In this paper, using T_W-base fuzzy arithmetic operations, we have derived a new model in two forms for the general fuzzy time series which indicate that the value of $F(t)$ at $t+1$ can be related to its previous ones by means of forward and backword linguistic difference of various orders. This proves the problem Song et al. [8] suggested in concluding remarks of their paper.

5. References

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