바이폴라 퍼지집합

Bipolar Fuzzy Sets

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요 약

퍼지 집합은 경계가 애매한 집단, 어떤 제약에 대한 만족정도가 애매한 개체들의 모임, 또는 애매한 개념을 소속정도를 이용하여 표현한다. 퍼지 집합에서는 자신의 나타내는 개념이나 제약에 대해서 무관한 개체나 상반되는 개체에 대해서도 소속정도 값으로 0을 부여한다. 응용에 따라서는 집합이 나타내는 개념이나 제약에 대해서 무관한 것과 상반되는 것을 구별하여 표현하는 것이 유용한 경우도 있다. 이 논문에서는 퍼지 집합에서 소속정도값 0을 갖는 무관한 원소들과 상반되는 원소들을 구별하여 표현하기 위해 소속정도값의 영역을 구간 [−1, 1]로 확장한 바이폴라 퍼지집합이라는 확장된 퍼지 집합을 소개한다. 한편, 바이폴라 퍼지 집합에 대한 집합연산, 퍼지정도 척도, 관계, 추론 등의 연산에 대해서도 소개한다.

1. Introduction[1]

Fuzzy sets are a kind of useful to represent a collection of elements with vague boundaries, a collection of elements satisfying fuzzy constraints, and fuzzy concepts by employing membership degree components. In fuzzy sets, the membership degrees of elements range over the interval [0,1]. The membership degree 1 indicates that element completely belongs to its corresponding set, and the membership degree 0 indicates that element does not belong to the set. The membership degree on the interval (0,1) indicates the partial membership degree to the set. That is, membership degree represents the degree of belonging of elements to set. Sometimes, it can represent the satisfaction degree of elements to some property or constraint corresponding to the set.[1,2]

In the viewpoint of satisfaction degree, the membership degree 0 indicates the situation at which elements do not satisfy the corresponding property. In the perspective of the membership degree, we cannot discriminate each other the elements with the membership degree 0. Suppose that there is a fuzzy set "*young*" defined on the *age* domain [0, 100] as shown in Figure 1. In this fuzzy set, both age 50 and 95 have the membership degree 0. Although they do not satisfy the property "*young*", we may say that age 95 is more apart from the property rather than age 50. Only with the membership degree ranged on the interval [0, 1], it is difficult to express this kind of differences.

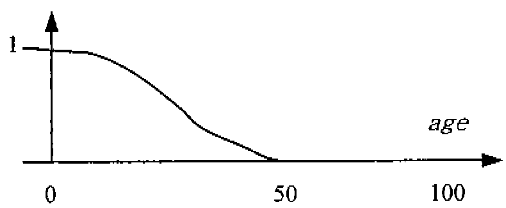

Figure 1: A fuzzy set "*young*"

Now consider another fuzzy set "*frog's prey*" like this:

frog's prey = {(*mosquito*, 1), (*dragon fly*, 0.4), (*turtle*, 0.0), (*snake*, 0.0)}

In this fuzzy set, *turtle* and *snake* have the membership degree 0. We know that *frog* and *turtle*

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[1] 이 연구는 첨단정보기술연구센터(AITrc)를 통해서 한국과학재단의 지원을 받아 수행된 것임.

are indifferent each other in the perspective of prey relationship, but *frog* is a prey of *snake*. As we can see in this example, it is difficult to express the difference of the irrelevance from the counter-relationship in fuzzy sets. If we can express this kind of difference in a set description, it comes to be more informative than traditional fuzzy set representation. In these observations, we introduce an extension of fuzzy sets named *bipolar fuzzy sets*.

This paper is organized as follows: Section 2 discusses the rationale to introduce bipolar fuzzy sets and presents their representation methods. Section 3 defines some set operations, a fuzzy measure and extension principles for bipolar fuzzy sets. Section 4 describes bipolar fuzzy relations and bipolar fuzzy. In final, Section 5 addresses conclusions and further study issues.

2. Bipolar Fuzzy Sets

Bipolar fuzzy sets have a membership function whose range is [−1, 1]. In bipolar fuzzy sets, the membership degree 0 indicates the irrelevance of element to the property implied by set, the membership degree in (0,1] indicates the satisfaction degree of element to the property, and the membership degree in [−1,0) indicates the satisfaction degree of element to the contradictory-property for set. Figure 2 shows a bipolar fuzzy set redefined for the fuzzy set "*young*" of Figure 1. In this representation, the membership degrees for age 50 and 95 are different each other and provide more information about the satisfaction degree to the property "*young*".

One may think that if we map the membership degree range [−1,1] of bipolar fuzzy sets to the range [0,1] of traditional fuzzy sets, we may have the same effect. Suppose that we translate and scale the membership function for "*young*" of Figure 2 onto the range [0,1]. Then we come to have a membership function shown in Figure 3. As we can see in the membership function, the satisfaction degree of *age* to the property "*young*" is not equal to 0 until *age* becomes 100. It is not what we want to express in the set.

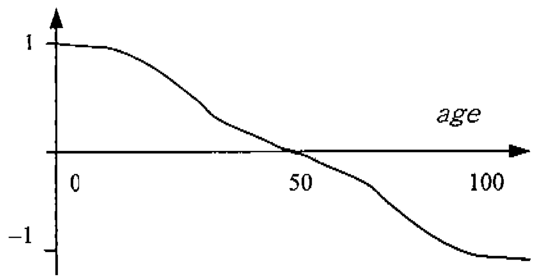

Figure 2: A bipolar fuzzy set "*young*"

The fuzzy set "*frog's prey*" can be redefined by a bipolar fuzzy set as follows:

frog's prey = {(*mosquito*, 1), (*dragon fly*, 0.4), (*turtle*, 0.0), (*snake*,−1)}

In the bipolar fuzzy set, the membership degrees of *turtle* and *snake* are different. The membership degree 0 of *turtle* means that *frogs* do not hunt *turtle*. The membership degree −1 of *snake* means that *frogs* are hunted by *snakes*. We can see that the bipolar fuzzy set representation can be more informative than the traditional fuzzy set representation in some situations.

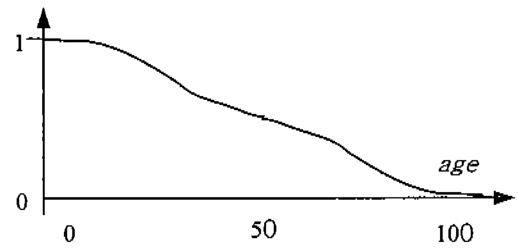

Figure 3: A fuzzy set corresponding to bipolar fuzzy set "*young*" in Figure 2

In bipolar fuzzy sets, we use two kinds of representation methods: canonical representa-tion and reduced representation. In the canonical representation method, a pair of numbers (*positive membership degree*, *negative membership degree*) represent membership degrees. That is, the membership degrees are divided into two parts: positive part in [0, 1] and negative part in [−1, 0]. In the reduced representation method, membership degrees are represented by a number in [−1, 1]. The followings give the definitions for those representation methods.

Def. **Canonical representation of bipolar fuzzy set** A on the domain X

$$\mu_A : X \to [0,1]\times[-1,0]$$

$$\mu_A(x) = (\mu_A^P(x), \mu_A^N(x))$$

The positive membership degree $\mu_A^P(x)$ denotes the satisfaction degree of element x to the constraint corresponding to A, and the negative membership degree $\mu_A^N(x)$ denotes the satisfaction degree of x to the contradictory-constraint of A. If $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) = 0$, it is the situation that x is decided to have only positive satisfaction for A. If $\mu_A^P(x) = 0$ and $\mu_A^N(x) \neq 0$, it is the situation that x is decided to have only negative satisfaction for A. In canonical representation, it is possible to be $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) \neq 0$. It is the situation in which some confusing decision has been made for the elements.

The following shows the canonical

representation for the above bipolar fuzzy set *frog's prey*.

frog's prey = {(*mosquito*, (1,0)), (*dragon fly*, (0.4,0)), (*turtle*, (0,0)), (*snake*, (0,−1))}

Def. **Reduced representation of bipolar fuzzy set** A on the domain X

$$\mu_A^r : X \to [-1, 1]$$

The membership degree $\mu_A^r(x)$ of reduced representation of bipolar fuzzy set A can be induced from its canonical representation as follows:

$$\mu_A^r(x) = \begin{cases} \mu_A^P(x) & \text{if } \mu_A^N(x) = 0 \\ \mu_A^N(x) & \text{if } \mu_A^P(x) = 0 \\ f(\mu_A^P(x), \mu_A^N(x)) & \text{otherwise} \end{cases}$$

For the cases in which both $\mu_A^P(x)$ and $\mu_A^N(x)$ have non-zero values, the aggregation function $f(\mu_A^P(x), \mu_A^N(x))$ can be defined in various ways. The choice of the aggregation function depends on the application domains. The next shows two candidates for the aggregation function.

$$f(\mu_A^P(x), \mu_A^N(x)) = \mu_A^P(x) + \mu_A^N(x)$$

$$f(\mu_A^P(x), \mu_A^N(x)) = \text{sgn}(\mu_A^P(x), \mu_A^N(x)) \times \max\{\mu_A^P(x)(1 + \mu_A^N(x)), -\mu_A^N(x)(1 - \mu_A^P(x))\}$$

Here $\text{sgn}(\mu_A^P(x), \mu_A^N(x))$ denotes the sign of the number whose absolute value is larger. That is,

$$\text{sgn}(\mu_A^P(x), \mu_A^N(x)) = \begin{cases} 1 & \text{if } \mu_A^P(x) \ge |\mu_A^N(x)| \\ -1 & \text{otherwise} \end{cases}$$

The reduced representation seems to be easy to use rather than canonical representation. In this paper, we use the canonical representation to define operations on bipolar fuzzy sets since it allows more natural extension of fuzzy set theory to bipolar fuzzy set theory.

3. Operations of Bipolar Fuzzy Sets

Bipolar fuzzy sets can be considered as an extension of fuzzy sets. However, set operations for bipolar fuzzy sets should be redefined since the range of their membership function is different from that of traditional membership function.

This section gives new definitions of set operations, fuzziness measure, and extension principle for bipolar fuzzy sets. Suppose that there are two bipolar fuzzy sets A and B whose membership functions are as follows. Here X denotes the universe of discourse.

$$\mu_A(x) = (\mu_A^P(x), \mu_A^N(x))$$

$$\mu_B(x) = (\mu_B^P(x), \mu_B^N(x))$$

The following presents several new set operations defined for bipolar fuzzy sets. It is assumed that bipolar fuzzy sets are expressed in the canonical representation. The set operations are developed to handle fuzzy sets as a special case of bipolar fuzzy sets.

Def. **Union** $A \cup B$ of A and B

$$A \cup B = \{(x, \mu_{A\cup B}(x)) \mid x \in X\}$$

$$\mu_{A\cup B}(x) = (\mu_{A\cup B}^P(x), \mu_{A\cup B}^N(x))$$

$$\mu_{A\cup B}^P(x) = \max\{\mu_A^P(x), \mu_B^P(x)\}$$

$$\mu_{A\cup B}^N(x) = \min\{\mu_A^N(x), \mu_B^N(x)\}$$

Def. **Intersection** $A \cap B$ of A and B

$$A \cap B = \{(x, \mu_{A\cap B}(x)) \mid x \in X\}$$

$$\mu_{A\cap B}(x) = (\mu_{A\cap B}^P(x), \mu_{A\cap B}^N(x))$$

$$\mu_{A\cap B}^P(x) = \min\{\mu_A^P(x), \mu_B^P(x)\}$$

$$\mu_{A\cap B}^N(x) = \max\{\mu_A^N(x), \mu_B^N(x)\}$$

Def. **Complement** $\overline{A}$ of A

$$\overline{A} = \{(x, \mu_{\overline{A}}(x)) \mid x \in X\}$$

$$\mu_{\overline{A}}(x) = (\mu_{\overline{A}}^P(x), \mu_{\overline{A}}^N(x))$$

$$\mu_{\overline{A}}^P(x) = 1 - \mu_A^P(x)$$

$$\mu_{\overline{A}}^N(x) = -1 - \mu_A^N(x)$$

Def. **Subset** $A \subset B$

$A \subset B$ if and only if $\mu_A^P(x) \le \mu_B^P(x)$ and $\mu_A^N(x) \ge \mu_B^N(x)$ for all $x \in X$.

Def. **Cardinality** $Card(A)$ of A

$$Card(A) = \sum_{x\in X} (\mu_A^P(x) + |\mu_A^N(x)|)$$

Def. **α-cut** A_α of A

$$A_\alpha = A_\alpha^P \cup A_\alpha^N$$

$$A_\alpha^P = \{x \mid \mu_\alpha^P(x) \ge \alpha\}$$

$$A_\alpha^N = \{x \mid \mu_\alpha^N(x) \le -\alpha\}$$

We call A_α^P as positive α-cut and A_α^N as negative α-cut.

Def. **Support** *Supp*(A) of A

$$Supp(A) = Supp^P(A) \cup Supp^N(A)$$
$$Supp^P(A) = \{x \mid \mu_A^P(x) > 0\}$$
$$Supp^N(A) = \{x \mid \mu_A^N(x) < 0\}$$

We call $Supp^P(A)$ as positive support and $Supp^N(A)$ as negative support.

The following is a fuzziness measure that can be applied to bipolar fuzzy sets.

Def. **Fuzziness measure** $Fz(A)$

$$Fz(A) = E_i 2^{1-(\Delta_i/M_i)}$$

$$E_i = -\sum_i \mu_A^P(x_i)\log_2\mu_A^P(x_i) - \sum_i |\mu_A^N(x_i)|\log_2|\mu_A^N(x_i)|$$

$$\Delta_i = |\mu_A^P(x_i) + \mu_A^N(x_i)|$$

$$M_i = \max\{\mu_A^P(x_i), |\mu_A^N(x_i)|\}$$

In bipolar fuzzy sets, we consider two aspects of fuzziness. One is the respective fuzziness of $\mu_A^P(x)$ and $\mu_A^N(x)$ of which fuzziness increases as much as its membership degree approaches to 0.5 and –0.5, respectively. This quantity of fuzziness is measured by the equation E_i which takes after the Entrophy measure[2] of fuzzy sets. The other is the fuzziness incurred in the situation that an element has both positive and negative membership degrees. The quantity of fuzziness is measured by $2^{1-(\Delta_i/M_i)}$ which produces larger number as long as the magnitudes of $\mu_A^P(x)$ and $\mu_A^N(x)$ are more similar and larger simultaneously.

The extension principle is one of the most basic concepts of fuzzy set theory which can be used to generalize crisp mathematical concepts to fuzzy sets. In the line of the same objective, the extension principle for bipolar fuzzy sets is defined as follows:

Def. **Extension Principle**

Let X be a Cartesian product of universes $X = X_1 \times X_2 \times \Lambda \times X_n$ and A_1, Λ, A_n be n bipolar fuzzy sets in X_1, Λ, X_n, respectively. f is a mapping from X to a universe Y, $y = f(x_1, K, x_n)$. Then a bipolar fuzzy set B in Y is defined as follows:

$$B = \{(y, (\mu_B^P(y), \mu_B^N(y))) \mid y = f(x_1, K, x_n), (x_1, K, x_n) \in X_1 \times X_2 \times \Lambda \times X_n\}$$

$$\mu_B^P(y) = \begin{cases} \max_{(x_1,K,x_n)\in f^{-1}(y)} \min_i \mu_{A_i}^P(x) & if\ f^{-1}(y) \neq \phi \\ 0 & otherwise \end{cases}$$

$$\mu_B^N(y) = \begin{cases} \min_{(x_1,K,x_n)\in f^{-1}(y)} \max_i \mu_{A_i}^N(x) & if\ f^{-1}(y) \neq \phi \\ 0 & otherwise \end{cases}$$

4. Bipolar Fuzzy Relation and Bipolar Fuzzy Inference

This section presents a method to induce bipolar fuzzy relation from a pair of bipolar fuzzy sets and a bipolar fuzzy inference method that uses bipolar fuzzy relation.

Bipolar fuzzy relation is defined as follows: Here X and Y are universes.

Def. **Bipolar fuzzy relation** R on $X \times Y$

$$R = \{((x, y), \mu_R(x, y)) \mid x \in X, y \in Y\}$$
$$\mu_R(x, y) = (\mu_R^P(x, y), \mu_R^N(x, y))$$
$$\mu_R^P(x, y) \in [0,1],\ \ \mu_R^N(x, y) \in [-1,0]$$

In fuzzy theory, fuzzy relations are constructed to associate two fuzzy sets, especially in fuzzy rules like $A \to B$. The followings is a method to build bipolar fuzzy relation $R(A,B)$ from two bipolar fuzzy sets A and B. Here the universe of A is X and that of B is Y.

Def. **Bipolar fuzzy relation** $R(A,B)$ for bipolar fuzzy sets A and B

$$R(A,B) = \{((x, y)\mu_{R(A,B)}(x, y)) \mid x \in X, y \in Y,$$
$$\mu_{R(A,B)}(x, y) = (\mu_{R(A,B)}^P(x, y), \mu_{R(A,B)}^N(x, y))\}$$
$$\mu_{R(A,B)}^P(x, y) = \min\{\mu_A^P(x), \mu_B^P(y)\}$$
$$\mu_{R(A,B)}^N(x, y) = \max\{\mu_A^N(x), \mu_B^N(y)\}$$

Suppose that there are two bipolar fuzzy sets like this:

A = {(a, (1,-0.2)), (b, (0.4,-0.1)), (c, (0.1,0)), (d, (0.3,-0.6)), (e, (0.5,-0.8))}

B = {(α, (0,-0.5)), (β, (0,-0.3)), (γ, (0,0)), (δ, (0.5,0)), (ε, (1.0,0))}

Table 1 is a bipolar fuzzy relation induced from A and B.

Table 1. A bipolar fuzzy relation

	α	β	γ	δ	ε
a	(0.0,0.2)	(0.0,-0.2)	(0.0,0 0)	(0.5,0.0)	(1.0,0.0)
b	(0.0,-0.1)	(0.0,-0.1)	(0.0,0.0)	(0.4,0.0)	(0.4,0.0)
c	(0.0, 0.0)	(0.0, 0.0)	(0.0,0.0)	(0.1, 0.0)	(0.1, 0.0)
d	(0.0,-0.5)	(0.0,-0.3)	(0.0,0.0)	(0.3,0.0)	(0.3,0.0)
e	(0.0,-0.5)	(0.0,-0.3)	(0.0,0.0)	(0.5,0.0)	(0.5,0.0)

Fuzzy inference is a very useful mechanism to deduce new information from rules and facts expressed with fuzzy sets. Many successful applications of fuzzy inference have been reported especially in control, knowledge-based systems, and so on.[4]

The followings show the bipolar fuzzy inference method based on the bipolar fuzzy relation which is induced from bipolar fuzzy rules. Here R is a bipolar fuzzy relation for a bipolar fuzzy rule $A \rightarrow B$ where A and B are bipolar fuzzy sets. A' is an input bipolar fuzzy set and B' is the inference result expressed in a bipolar fuzzy set. The universe of A and A' is X and that of B and B' is Y.

Def. **Composition Rule of Inference** for bipolar fuzzy sets $A' \circ R$

$$B' = A' \circ R = \{(y, \mu_{A' \circ R}(y)) \mid y \in Y\}$$

$$\mu^P_{A' \circ R}(y) = \max_x \min\{\mu^P_{A'}(x), \mu^P_R(x,y)\}$$

$$\mu^N_{A' \circ R}(y) = \min_x \max\{\mu^N_{A'}(x), \mu^N_R(x,y)\}$$

Suppose that a bipolar fuzzy relation R for a fuzzy rule is like Table 1. The followings show the fuzzy inference results B' for bipolar fuzzy input A' with respect to R.

A' = { (a, (0.0,0.0)), (b, (0.3,0.0)), (c, (0.0,0.0)), (d, (0.0,0.0)), (e, (0.0,0.0))}
B' = { (α, (0.0,0.0)), (β, (0.0,0.0)), (γ, (0.0,0.0)), (δ, (0.3,0.0)), (ε, (0.3,0.0))}

A' = { (a, (0.3,0.0)), (b, (0.0,0.0)), (c, (0.0,0.0)), (d, (0.0,-0.4)), (e, (0.0,0.0))}
B' = { (α, (0.0,-0.4)), (β, (0.0,-0.3)), (γ, (0.0,0.0)), (δ, (0.3,0.0)), (ε, (0.3,0.0))}

A' = { (a, (0.0,0.0)), (b, (0.0,0.0)), (c, (0.0,0.0)), (d, (0.5,-0.2)), (e, (0.0,0.0))}
B' = { (α, (0.0,-0.2)), (β, (0.0,-0.2)), (γ, (0.0,0.0)), (δ, (0.3,0.0)), (ε, (0.3,0.0))}

5. Conclusions

In this paper, we proposed a new extension of fuzzy sets named *bipolar fuzzy sets*. Bipolar fuzzy sets are developed of which membership functions have their range in [−1, 1]. Bipolar fuzzy sets make it possible to discriminate irrelevant elements from contradictory elements, to which membership degree 0 is assigned in traditional fuzzy sets. Thus we can express fuzzy knowledge more flexibly with bipolar fuzzy sets.

We introduced new set operations, fuzziness measures, and extension principle for bipolar fuzzy sets. Since traditional fuzzy sets can be regarded as a special case of bipolar fuzzy sets, we defined the new definitions to be applicable to traditional fuzzy sets as well as bipolar fuzzy sets.

In addition, we presented a bipolar fuzzy inference method which can be applied to bipolar fuzzy knowledge base consisting of fuzzy rules and bipolar fuzzy input. Since a bipolar fuzzy set can express both a concept and its contradictory concept simultaneously, bipolar fuzzy sets can express fuzzy knowledge in a more simpler way rather than traditional fuzzy sets.

This study on bipolar fuzzy sets is a preliminary one. Further studies are needed especially in mathematical properties and semantic interpretation on bipolar fuzzy sets and bipolar fuzzy inference methods.

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