퍼지 기대값에 대한 일치추정량

Consistent Estimator for the Fuzzy Expectation

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Abstract

In this paper, we show that the fuzzy sample mean is a strong consistent estimator for the expectation of a fuzzy random set taking values in the space $F(R^p)$ of upper semicontinuous convex fuzzy subsets of R^p with compact support.

1. Introduction

Since Puri and Ralescu [11] introduced the concept of fuzzy random variables, there has been increasing interest in statistical inference for fuzzy stochastic model. Schnatter [12] introduced the concept of fuzzy sample mean and fuzzy sample variance in order to discuss the generalization of statistical methods to fuzzy data. Yao and Hwang [13] studied point estimation for random sample with one vague data. Recently, Grzegorzewski [4] proposed a definition of fuzzy test for testing statistical hypotheses with vague data and Korner [9] also suggested a method to test hypotheses about the expectation of a fuzzy random variable.

The purpose of this paper is to show that the fuzzy sample mean is a strong consistent estimator for the expectation of a fuzzy random variable taking values in the space $F(R^p)$ of upper semicontinuous convex fuzzy subsets of R^p with compact support. To this end, strong laws of large numbers (for short, SLLN) for fuzzy random variables should be considered. The SLLN for fuzzy random variables was obtained by Klement et al. [8], Joo and Kim [7], Molchanov [10], and etc. Our result generalizes the results of earlier works to the case of a more general setting.

2. Preliminaries

Let $K(R^p)$ denote the family of non-empty compact convex subsets of the Euclidean space R^p. Then the space $K(R^p)$ is metrizable by the Hausdorff metric defined by

$$h(A,B) \;=\; \max\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\}.$$

A norm of $A\in K(R^p)$ is defined by $\|A\| = h(A,\{a\}) = \sup_{a\in A}|a|$. The addition and scalar multiplication on $K(R^p)$ are defined as usual:

$$A\oplus B = \{a+b : a\in A, b\in B\},\quad \lambda A = \{\lambda a : a\in A\}$$

for $A,B\in K(R^p)$ and $\lambda\in R$.

Throughout this paper, let (Ω,Σ,P) be a probability space. A set-valued function $X:\Omega\to K(R^p)$ is called measurable if for each closed subset B of R^p,

$$X^{-1}(B) \;=\; \{\omega : X(\omega)\cap B \neq \emptyset\}$$

is a measurable set. It is well-known that the measurability of X is equivalent to the measurability of X considered as a map from Ω to the metric space $K(R^p)$ endowed with the Hausdorff metric h. A set-valued function $X:\Omega\to K(R^p)$ is called a random set if it is measurable. A random set is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set X is defined by

$$E(X) \;=\; \{Ef : f\in L(\Omega,R^p) \text{ and } f(\omega)\in X(\omega)\ a.s.\}.$$

The following SLLN for random sets was proved by Artstein and Vitale [1].

Theorem 2.1. Let $\{X_n\}$ be a sequence of independent and identically distributed random sets. If $E\|X_1\| < \infty$, then

$$\lim_{n\to\infty} h\left(\frac{1}{n}\oplus_{i=1}^{n}X_i,\ EX_1\right) \;=\; 0 \quad a.s.$$

3. Main Results

Let $F(R^p)$ denote the family of all fuzzy sets $u : R^p \to [0,1]$ with the following properties;

(1) u is normal, i.e., there exists $x\in R^p$ such that $u(x)=1$;

(2) u is upper semicontinuous;

(3) u is a convex fuzzy set, i.e., $u(\lambda x+(1-\lambda)y) \geq \min(u(x),u(y))$ for $x,y\in R^p$ and $\lambda\in[0,1]$;

(4) $\operatorname{supp} u \;=\; \overline{\{x\in R^p : u(x)>0\}}$ is compact.

For a fuzzy set u in R^p, the α-level set of u is defined by

$$L_\alpha u = \begin{cases} \{x : u(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1, \\ \text{supp } u & \text{if } \alpha = 0. \end{cases}$$

Then, it follows immediately that $u \in F(R^p)$ if and only if $L_\alpha u \in K(R^p)$ for each $\alpha \in [0,1]$. The linear structure on $F(R^p)$ is defined as usual;

$$(u \oplus v)(z) = \sup_{x+y=z} \min(u(x), v(y)),$$

$$(\lambda u)(z) = \begin{cases} u(z/\lambda) & \text{if } \lambda \neq 0, \\ I_{\{0\}} & \text{if } \lambda = 0, \end{cases}$$

for $u, v \in F(R^p)$ and $\lambda \in R$, where $I_{\{0\}}$ is the indicator function of $\{0\}$.

Lemma 3.1. For $u \in F(R^p)$, we define

$$f_u : [0,1] \to (K(R^p), h), \quad f_u(\alpha) = L_\alpha u.$$

Then the followings hold;

(1) f_u is left continuous on $(0,1]$,

(2) f_u has right-limits on $(0,1]$ and f_u is right-continuous at 0.

We denote $\overline{\bigcup_{\beta > \alpha} L_\beta u}$ by $L_{\alpha^+} u$. Then the right limit of f_u at α is $L_{\alpha^+} u$. Now we define, for $J \subset [0,1]$,

$$w_u(J) = \sup_{\alpha_1, \alpha_2 \in J} h(L_{\alpha_1} u, L_{\alpha_2} u)$$

then it follows that for $0 \leq \alpha < \beta \leq 1$,

$$w_u(\alpha, \beta) = w_u(\alpha, \beta] = h(L_{\alpha^+} u, L_\beta u),$$

and

$$w_u[\alpha, \beta) = w_u[\alpha, \beta] = h(L_\alpha u, L_\beta u).$$

Lemma 3.2. For each $u \in F(R^p)$ and $\varepsilon > 0$, there exist a partition $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ of $[0,1]$ such that $w_u(\alpha_{i-1}, \alpha_i) < \varepsilon$, $i = 1, 2, \ldots, r$.

Now, in order to generalize the Hausdorff metric on $K(R^p)$ to $F(R^p)$, we define the two metrics d_1, d_∞ on $F(R^p)$ by

$$d_1(u, v) = \int_0^1 h(L_\alpha u, L_\alpha v) \, d\alpha$$

$$d_\infty(u, v) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha u, L_\alpha v)$$

Also, the norm of u is defined as $\| u \| = d_\infty(u, I_{\{0\}}) = \sup_{x \in L_0 u} | x |$.

A fuzzy set valued function $X:\Omega \to F(R^p)$ is called measurable if for each closed subset B of R^p,

$$X^{-1}(B)(\omega) = \sup_{x\in B} X(\omega)(x)$$

is measurable when considered as a function from Ω to $[0,1]$. It is well-known that X is measurable if and only if for each $\alpha \in [0,1]$, $L_\alpha X$ is measurable as a set-valued function. A fuzzy set valued function $X:\Omega \to F(R^p)$ is called a fuzzy random set if it is measurable. A fuzzy random set X is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded fuzzy random set X is a fuzzy subset of R^p defined by $E(X)(x) = \sup\{\alpha\in[0,1]:\ x\in E(L_\alpha X)\}$.

The fuzzy random sets $X_1, X_2, \ldots, X_n$ are called independent if for every closed subsets $B_1, B_2, \ldots, B_n$ of R^p, the random variables $X_1^{-1}(B_1)$, $X_2^{-1}(B_2), \ldots, X_n^{-1}(B_n)$ are independent in the usual sense. Then it follows that $X_1, X_2, \ldots, X_n$ are independent if and only if the Borel σ-fields $\sigma\{L_\alpha X_1: \alpha\in[0,1]\}$, $\sigma\{L_\alpha X_2: \alpha\in[0,1]\}$, $\ldots$, $\sigma\{L_\alpha X_n: \alpha\in[0,1]\}$ are independent in the usual sense. Also, the fuzzy random sets $X_1, X_2, \ldots, X_n$ are said to have the same fuzzy distribution as X if for every closed subsets B of R^p, the random variables $X_1^{-1}(B), X_2^{-1}(B), \ldots, X_n^{-1}(B)$ have the same distribution as $X^{-1}(B)$ in the usual sense. The fuzzy random sets $X_1, X_2, \ldots, X_n$ are called a fuzzy random sample from the population with fuzzy distribution as a fuzzy random sets X if they are independent and have the same fuzzy distribution as X. For a fuzzy random sample $X_1, X_2, \ldots, X_n$, the fuzzy sample mean is defined by

$$\overline{X_n} = \frac{1}{n} \oplus_{i=1}^{n} X_i.$$

It follows that the fuzzy random sample is an unbiased estimator, i.e., $E(\overline{X_n}) = E(X)$. A strong law of large numbers by Klement et al.[8] implies that the fuzzy sample mean $\overline{X_n}$ is a strong consistent estimator for the fuzzy expectation $E(X)$ with respect to the metric d_1. The next theorem shows that the fuzzy sample mean $\overline{X_n}$ is a strong consistent estimator for the fuzzy expectation $E(X)$ with respect to the metric d_∞.

Theorem 3.4. Let $\{X_n\}$ be a fuzzy random sample from the population with fuzzy distribution as a fuzzy random variable X. If $E\|X\| < \infty$, then

$$\lim_{n\to\infty} d_\infty(\overline{X_n}, EX) = 0 \quad a.s.$$

Example 1. Let $u \in F(R^p)$ be fixed and $X(\omega) = u(x - Y(\omega))$ be a fuzzy random set with the same fuzzy distribution as the population, where Y is a random vector taking values in R^p with $E|Y| < \infty$. Since $L_\alpha X(\omega) = Y(\omega) + L_\alpha u$, we have $E(L_\alpha X) = EY + L_\alpha u$. Hence, $E(X)(x) = u(x - EY)$. Now if $\{Y_n\}$ is a random sample from the population with distribution of Y and $X_n(\omega) = u(x - Y_n(\omega))$, then $\{X_n\}$ is a fuzzy random sample from the population with fuzzy distribution of X, and the fuzzy sample mean is $\overline{X_n} = u(x - \overline{Y_n})$, where $\overline{Y_n} = \frac{1}{n} \sum_{i=1}^{n} Y_i$ is the usual sample mean of $Y_1, Y_2, \ldots, Y_n$. Hence, by the above theorem,

$$\lim_{n\to\infty} d_\infty(\overline{X_n}, E(X)) = \lim_{n\to\infty} d_\infty(u(x - \overline{Y_n}), u(x - EY)) = 0 \ a.s.,$$

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