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정규모우드의 안정성 변화에 따른 분기모우드의 계산법

박 철 희

Computations of bifurcating modes due to the stability change of normal modes

Chol-Hui Pak

ABSTRACT

It is shown, in nonlinear two-degree-of freedom system, that the bifurcating modes are created by the stability changes of normal modes. There are four types of stability criterion, each of which gives rise to a distinct functional form of bifurcating modes; the bifurcating mode is born in the form of eigenfunction through which the stability is changed. Then a procedure is formulated to compute the bifurcating mode by the method of harmonic balance. Application of bifurcating mode to forced vibrations is introduced.

1. Introduction

Consider a conservative system in which the kinetic energy K and potential energy V are written

$$K=\frac{1}{2}\sum_{ij=1}^{2} m_{ij}(x)\dot{x}_i\dot{x}_j,\quad V=V(x),\quad K+V=h \qquad (1)$$

where $x=(x_1, x_2)$ and $\dot{x}=(\dot{x}_1, \dot{x}_2)$ are the generalized coordinates and velocity, respectively. Assume

(i) $m_{ij}(-x)=m_{ij}(x)$ and $V(-x)=V(x)$, and they are smooth.

(ii) The configuration space $\Gamma(h)=\{x\in R^2 \mid h-V(x)\geq 0\}$ is a simple closed domain in R^2, containing the origin.

(iii) $\nabla V\neq 0$ on $\partial\Gamma(h)=\{x\in R^2 \mid h-V(x)=0\}$ which is assumed to be a simple closed curve.

Then the existence of normal mode is shown by Pak and Rosenberg(1968).

The stability of normal mode may be determined by the variational equations(two second order coupled differential equations) which are obtained by perturbing the equations of motion and linearing the resulting equations. Then four characteristic multipliers are computed. Due to the symplectic property of Hamiltonian, the product of each pair is unity. In particular, one pair is unit if the system is conservative; $\lambda_3=\lambda_4=1$. The remaining multipliers λ_1 and λ_2 with $\lambda_1\lambda_2=1$ determine the stability, implying that a normal mode is generically shown to be a center or a saddle in Poincare map. Then it is shown that the stable and unstable manifolds of saddle either form a homoclinic orbit or intersect transversally, in either case a center is born, called a bifurcating mode.

The functional form of bifurcating mode is determined by Pak(1999); a bifurcating mode is born in the form of eigenfunction corresponding to the transition curve of stability chart. To prove it, the calculus of variations is utilized, in particular the geometry of envelope which is created from $\partial \Gamma(h)$. Then the method of harmonic balance is formulated.

Examples are given to compute the bifurcating modes in elastically coupled and inertially coupled nonlinear systems. Then a procedure is introduced to study forced vibrations whose response may be periodic, quasi-periodic or chaotic.

2. The birth of bifurcating modes

Let $x^*(t)$ be a normal mode. To study the stability of $x^*(t)$, it is perturbed by $x = x^* + \eta$. By substituting in the equations of motion

$$\frac{d}{dt}\left(\sum_{j=1}^{2} m_{ij}(x)\dot{x}_j\right) - \frac{\partial}{\partial x_j}\left(\sum m_{ij}\dot{x}\dot{x}_k\right) + \frac{\partial V}{\partial x_i} = 0, \quad i=1,2 \qquad (2)$$

and by linearing, one obtains

$$A(t)\ddot{\eta} + B(t)\dot{\eta} + C(t)\eta = 0 \qquad (3)$$

where A, B and C are periodic of period T.

Then Floquet theory is applicable to compute the characteristic multipliers

$$\eta(t+T) = \lambda\,\eta(t), \text{ for all } t>0. \qquad (4)$$

There are four multipliers. But the product of each pair is unity, and one pair is unity if the system is conservative; $\lambda_1\lambda_2 = 1$ and $\lambda_3 = \lambda_4 = 1$ as shown in Fig. 1. Therefore, a normal mode is generically a center or a saddle, shown in Fig. 2, and the non-generic case corresponds to the stability change or coexistence.

By utilizing the uniqueness theorem applied to the equations of motion and by noting that a Hamiltonian system does not possess a limit set, we have

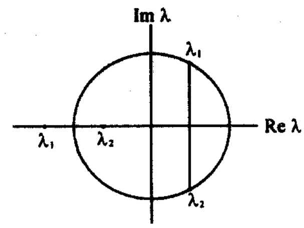


Fig. 1 Characteristic multiplier

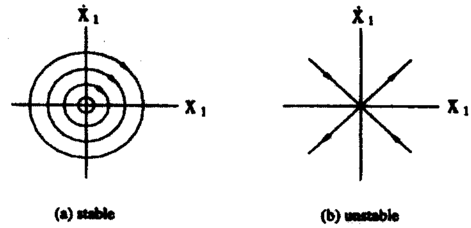
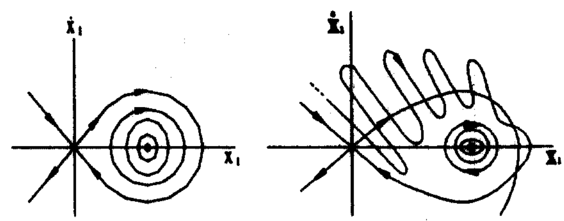


Fig. 2 Poincare map of normal mode

Lemm 1. The stable and unstable manifolds of saddle(unstable normal mode) either from a homoclinic orbit or intersect transversally, shown in Fig. 3. In either case, a new center is born, called a bifurcating mode.



(a) homoclinic orbit (b) transversal intersection

Fig. 3. The birth of bifurcating mode

Due to the symmetry assumption, a pair of bifurcating modes are born, implying that the stability change gives rise to a pitch-fork bifurcation.

3. The functional form of bifurcating modes

Synge(1926) derived a stability equation,

$$\ddot{\beta} + Q(t)\beta = 0 \qquad (5)$$

where β is the disturbance which is measured orthogonally from the normal mode, and $Q(t) = Kv^2 + 3x^2v^2 + \sum V_{ij} n_i n_j$

in which K is Gaussian curvature, v the speed of normal mode C^*, x the curvature of C^*, $V_{ij} = \nabla\nabla V$, and n_i the the unit vector orthogonal to C^*. Then a normal mode is said to be stable in the kinematic-statical sense if every solution to Eq (5) is bounded.

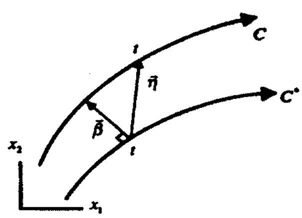


Fig. 4 The disturbance vector $\vec{\beta}$ ($\vec{\eta}$ is based on Liapunov sense)

Let $2T$ be the periodic of normal mode. Then the period of $Q(t)$ is T. Let $\beta_1(t)$ and $\beta_2(t)$ be the normalized solutions of Eq(5); $\beta_1(0)=1$, $\dot{\beta}_1(0)=0$ and $\beta_2(0)=0$, $\dot{\beta}_2(0)=1$. Then the characteristic multiplier λ satisfies,

$$\begin{aligned} &\lambda^2 - (\beta_1(T) + \dot{\beta}_2(T))\lambda + 1 = 0 \\ &\beta(t+T) = \lambda\beta(t) \end{aligned} \qquad (6)$$

Let us choose $t=0$ when the normal mode passes through the origin. Then $Q(-t) = Q(t)$. Due to Magnus and Winkler(1979), the following hold;

$$\begin{aligned} \beta_1(T) &= 2\beta_1(T/2)\dot{\beta}_2(T/2) - 1 \\ &= 2\dot{\beta}_1(T/2)\beta_2(T/2) + 1 \\ \dot{\beta}_2(T) &= \beta_1(T) \\ \dot{\beta}_1(T) &= 2\dot{\beta}_1(T/2)\beta_1(T/2) \\ \beta_2(T) &= 2\beta_2(T/2)\dot{\beta}_2(T/2) \end{aligned} \qquad (7)$$

Since the normal mode changes its stability at $\lambda=1$ or -1, the stability criterion is due to Eq (7)

$$\beta_i(T/2) = 0, \quad \dot{\beta}_1(T/2) = 0, \quad i = 1, 2. \qquad (8)$$

It is noted that at $t = T/2$, the normal mode arrives at the rest point. When $\lambda=1$ or -1, Eq (5) has periodic solutions, called the eigenfunctions, denoted bt $\beta^*(t)$ and the corresponding systems parameters the eigenvalues which present the transition curves in the stability chart.

Proposition2. Let $\beta^*(t)$ be an eigenfuction of stability equation for a normal mode in a system S. Assume that the normal mode starts from the origin of configuration space at $t=0$. Then there are four types of $\beta^*(t)$, expressed as follows;

Type 1 . $\dot{\beta}_2(T/2)=0$ if and only if $\beta^*(t)$ is odd and of period $2T$,

$$\beta^*(t) = \sum_{n=1}^{\infty} b_n \sin(2n-1)\omega t\ .$$

Type 2 . $\beta_2(T/2)=0$ if and only if $\beta^*(t)$ is odd and of period T,

$$\beta^*(t) = \sum_{n=1}^{\infty} b_n \sin 2n\omega t\ .$$

Type 3 . $\beta_1(T/2)=0$ if and only if $\beta^*(t)$ is even and of period $2T$,

$$\beta^*(t) = \sum_{n=1}^{\infty} a_n \cos(2n-1)\omega t\ .$$

Type 4 . $\dot{\beta}_1(T/2)=0$ if and only if $\beta^*(t)$ is even and of period T,

$$\beta^*(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\omega t\ .$$

for which the normal mode is expressed as

$$\begin{aligned} x_1(t) &= \sum_{n=1}^{\infty} A_n \sin(2n-1)\omega t \\ x_2(t) &= \sum_{n=1}^{\infty} B_n \sin(2n-1)\omega t \end{aligned} \qquad (9)$$

By utilizing the geometry of envelope which is crerated on $\partial\Gamma(h)$ and by showing that Eq(5) is identical to Jacobi differential equation, the following Theorems are proven by Pak(1999);

Theorem 3. If a normal mode in a system S changes its stability through the criterion $\beta_2(T/2)=0$ or $\dot{\beta}_2(T/2)=0$, then the bifurcating modes, described in Lemma. 1 are born in the form of eigenfunction.

Theorem 4. Assume that the potential energy $V(x)$ and the inertial coefficients $m_{ij}(x)$ are symmetric with respect to a similar normal mode in a system S. If the similar normal mode changes its stability through $\beta_1(T/2)=0$ or $\dot{\beta}_1(T/2)=0$, then the bifurcating modes are born in the form of eigenfunction.

4. Procedure of computations

Procedures are formulated to compute bifurcating modes by the method of harmonic balance. By assuming that normal modes are close to straight lines, let us transform the coordinate system such that the normal mode is on the new x-axis. Then the bifurcating mode may be computed by

$$x(t)=\sum_{j=1}^{N}A_j\sin(2j-1)\omega t$$
$$y(t)=c\beta^*(t) \tag{10}$$

in which c is a constant to be determined. Since normal modes and bifurcating modes are expressed in infinite series, the first few terms are used for the generating function to compute the bifurcating mode.

A generating function is said to be perfect if it is substituted in th equations of motion and balanced harmonically, then the resulting series are expressed in the form of generating function.

Proposition 5. The following generating function is perfect;

$$x(t)=\sum_{j=1}^{N}A_j\sin(2j-1)\omega t$$
$$y(t)=\sum_{j=1}^{N}B_j\sin(2j-1)\omega t \tag{11}$$

To prove it, Eq(11) is substituted into the equations of motion and harmonically balanced to obtain

$$\sum_{j=1}^{N}X_j(A,B,\omega)\sin(2j-1)\omega t=0$$
$$\sum_{j=1}^{N}Y_j(A,B,\omega)\sin(2j-1)\omega t=0 \tag{12}$$

Proposition 6. Assume that there are two similar normal modes, $x\equiv0$ and $y\equiv0$, in a system S which has cubic nonlinearity in elasticity and in inertia. If the $y\equiv0$ mode changes its stability, then the generating function is perfect for every stability criterion, $\beta_j(T/2)=0$ or $\dot{\beta}_j(T/2)=0$, $j=1,\ 2$. In words, the following generating functions are perfect;

for $\beta_1(T/2)=0$,

$$x(t)=\sum_{n=1}^{\infty}A_n\sin(2n-1)\omega t,$$
$$y(t)=\sum_{n=1}^{\infty}B_n\cos(2n-1)\omega t$$

for $\dot{\beta}_1(T/2)=0$,

$$x(t)=\sum_{n=1}^{\infty}A_n\sin(2n-1)\omega t$$
$$y(t)=\sum_{n=0}^{\infty}B_n\cos2n\omega t$$

for $\beta_2(T/2)=0$,

$$x(t)=\sum_{n=1}^{\infty}A_n\sin(2n-1)\omega t$$
$$y(t)=\sum_{n=1}^{\infty}B_n\sin2n\omega t$$

for $\dot{\beta}_2(T/2)=0$,

$$x(t)=\sum_{n=1}^{\infty}A_n\sin(2n-1)\omega t,$$
$$y(t)=\sum_{n=1}^{\infty}B_n\sin(2n-1)\omega t.$$

Similarly, if the $x\equiv0$ mode changes its stability, a perfect generating function is obtained by exchanging $x(t)$ and $y(t)$ of above equations.

Example 1. Given an elastically coupled nonlinear system

$$K=\frac{1}{2}(\dot{x}^2+\dot{y}^2)$$
$$V=\frac{1}{2}(x^2+p^2y^2)+ax^4+bx^2y^2+cy^4 \qquad (13)$$

the equations of motion are

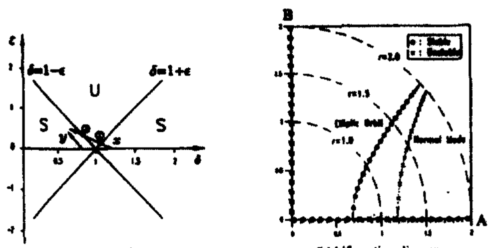
$$\ddot{x}+x+4ax^3+2bxy^2=0,\quad \ddot{y}+p^2y+2bx^2y+4cy^3=0.$$

There are two similar modes; the x-mode ($y\equiv0$) and y-mode($x\equiv0$), the x-mode is written as $x=A\cos\omega t$, $y=0$, $\omega^2=1+3aA^2$. Let $y=\beta$,

$$\frac{d^2\beta}{d\tau^2}+(\delta+2\varepsilon\cos 2\tau)\beta=0 \qquad (14)$$

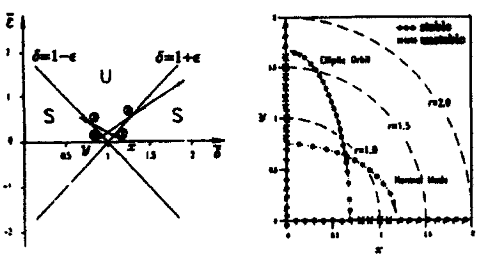
where $\tau=\omega t$ and

$$\delta=\frac{p^2}{\omega^2}+2\varepsilon,\ \varepsilon=\frac{bA^2}{2\omega^2}\ \text{or}\ \delta=p^2-(\frac{6a}{b}p^2-2)\varepsilon \qquad (15)$$



(a) stability chart (b) bifurcation diagram

Fig. 5 Bifurcation of the x-mode



(a) stability chart (b) bifurcation diagram

Fig. 6 Both the x- and y-mode changes the stability

For various parameters p, a, b and c, the stability of x-mode and y-mode, and the resulting bifurcating modes are shown in Fig 5 and 6.

Example 2. An inertially coupled nonlinear system is given by

$$K=\frac{1}{2}(1+\alpha y^2)\dot{x}^2+\frac{1}{2}\dot{y}^2,\quad V=\frac{1}{2}(x^2+\Omega^2y^2). \qquad (16)$$

The equations of motion are

$$(1+\alpha y^2)\ddot{x}+2\alpha y\dot{y}\dot{x}+\Omega^2x=0,$$
$$\ddot{y}-\alpha\dot{x}^2y+y=0$$

There are two similar modes; the x-mode ($y\equiv0$) and y-mode($x\equiv0$). The x-mode is written as $x=A\sin\Omega t$, $y=0$. Let $y=\beta$. Then

$$\frac{d^2\beta}{d\tau^2}+(\delta+2\varepsilon\cos 2\tau)\beta=0 \qquad (17)$$

where $\delta=1/\Omega^2+2\varepsilon$, $\varepsilon=-1/4\alpha A^2$, $\tau=\Omega t$. Similarly, the y-mode is written as $x=0$, $y=B\sin t$. By letting $x=\beta$, one obtains

$$[1+\varepsilon(1-\cos 2t)]\ddot{\beta}+2\varepsilon(\sin 2t)\dot{\beta}+\Omega^2\beta=0. \qquad (18)$$

Then the stability charts are shown in Fig. 7 and 8.

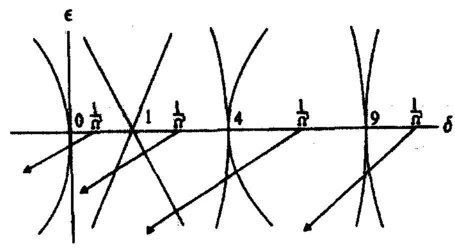


Fig. 7 Stability chart for x-mode

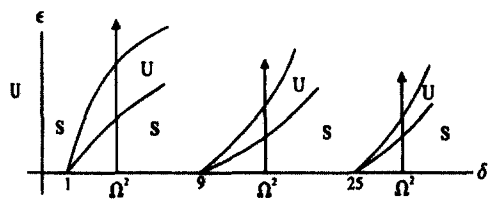


Fig. 8 Stability chart for y-mode

For $\Omega > 1$, the x-mode becomes unstable when the the motion curve intercepts the transition curve $\delta = -1/2\epsilon^2$ whose eigenfunction $\beta^*(t) = 1 - 1/2\epsilon\cos 2\Omega t$. Then the generating function is given by

$$x = A\sin\omega t, \quad y = B + C\cos 2\omega t \qquad (19)$$

By substituting into the equations of motion, one obtains

$$\begin{aligned} &A[\Omega^2 - \omega^2 - a\omega^2(B^2 + BC + \frac{1}{2}C^2)] = 0 \\ &B - \frac{1}{2}a\omega^2 A^2(B + \frac{1}{2}C) = 0 \qquad (20), \\ &(1 - 4\omega^2)C - \frac{1}{2}aA^2(B + C) = 0 \end{aligned}$$

For a given A, B, C and ω^2 are computed to construct the backbone curves.

A forcing system is said to be a natural forcing function if it is in the form of generating function in which each coefficient is arbitrary chosen and small. Then every single-mode excitation is a natural forcing function. As an example, the natural forcing function for the bifurcation mode given by Eq(19) is $f_x(t) = F_1\sin\omega t$, $f_y = F_0 + F_2\cos 2\omega t$. If the system(16) is excited by this natural forcing function, then the forced response is computed by Eq(20) in which the right-hand sides are replaced by F_1, F_0 and F_2, respectively. Then the forcing response is close to the bifurcating mode

If the initial condition is chosen in the neighborhood of the unstable x-mode and if the forcing system is $f_x(t) = F_1\sin\omega t$, $f_y(t) = 0$ with relatively large F_1, then the forced response may be quasi-periodic or chaotic.

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