Regression Quantile Estimators of a Nonlinear Time Series Regression Model

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ABSTRACT. In this paper, we deal with the asymptotic properties of the regression quantile estimators in the nonlinear time series regression model. For the sinusodial model which frequently appears for a time series analysis, we study the strong consistency and asymptotic normality of regression quantile estimators.

KEY WORDS : Regression Quantile Estimators; Strong Consistency; Asymptotic Normality.

1. Introduction

Generally, the nonlinear regression model is

$$y_t = f(x_t, \theta_0) + \epsilon_t, \qquad t = 1, 2, \cdots, T$$

where $f(x_t, \theta_0)$ is a real valued nonlinear function defined on R^{p+q}, x_t is a $(1 \times q)$ observed vector, the error term ϵ_t are independent and identically distributed(i.i.d.) with zero mean and finite variance. The parameter vector θ_0 which is interior point in a compact parameter space Θ is unknown and to be estimated. Jennrich(1969) first rigorously proved the existence of the least squares estimator(LSE) and showed the consistency and asymptotic normality of the LSE $\breve{\theta}_T$ under the several assumptions. Wu (1981) gave some sufficient conditions such as a Lipschitz type condition on the sequence $f(x_t, \theta)$ to prove the asymptotic properties of LSE $\breve{\theta}_T$.

The concept of periodicity in time series is of fundermental interest, since it provides a means for formalizing the notions of dependence or correlation between adjacent points. In this paper we think about a sum of sinusoidal components :

$$f(x_t, \theta_o) = \sum_{r=1}^{q} \{A_{ro}\cos(\omega_{ro}t) + B_{ro}\sin(\omega_{ro}t)\},$$

where $\theta_o = (A_{1o}, B_{1o}, \omega_{1o}, \cdots, A_{qo}, B_{qo}, \omega_{qo})$, for $q \geq 1, A_{ro}, B_{ro}$'s are some fixed unknown constants, ω_{ro} is unknown frequency lying between 0 to π $(1 \leq r \leq q)$ and in this case the observed value x_t means t. But the above formula does not satisfy Jennrich(1969)'s assumption nor Wu's Lipschitz type condition, the previous method to gain the LSE is not available. Walker(1971) obtained the asymptotic properties of an approximate LSE. Kundu(1993) and Kundu and Mitra(1996) gave the direct proof of consistency of the LSE and the asymptotic normality results and observed that the approximate LSE and the LSE are asymptotically equal. They found out $P(\breve{\theta}_T) = \left(P_1(\breve{\theta}_{1T}), P_2(\breve{\theta}_{2T}), \cdots, P_q(\breve{\theta}_{qT})\right)_{3q\times 1}$ converges in law $N(0,\ \sigma^2\Sigma^{-1})$, where σ^2 is the common variance of errors in the above model and $P_r(\breve{\theta}_{rT}) = \left(\sqrt{T}(\breve{A}_{rT} - A_{ro}), \sqrt{T}(\breve{B}_{rT} - B_{ro}), \sqrt{T}^3(\breve{\omega}_{rT} - \omega_{ro})\right)(1 \leq r \leq q)$, and Σ is defined in Theorem 4.1.

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In this paper we study the regression quantile estimators(RQE) which are defined in (1.2) of the following nonlinear time series model with assumptions A and B,

$$y_t = \sum_{r=1}^{q} [A_{ro}\cos(\omega_{ro}t) + B_{ro}\sin(\omega_{ro}t)] + \epsilon_t. \tag{1.1}$$

The RQE of the true parameter $\theta_o = (A_{1o}, B_{1o}, \omega_{1o}, \cdots, A_{qo}, B_{qo}, \omega_{qo})$ denoted by $\hat{\theta}_T = (\hat{A}_{1T}, \hat{B}_{1T}, \hat{\omega}_{1T}, \cdots, \hat{A}_{qT}, \hat{B}_{qT}, \hat{\omega}_{qT})$ is a parameter which minimizes the objective function

$$Q_T(\theta;\beta) = \frac{1}{T}\sum_{t=1}^{T} \varphi_\beta\Big(y_t - \sum_{r=1}^{q}[A_r\cos(\omega_r t) + B_r\sin(\omega_r t)]\Big), \tag{1.2}$$

where $0 < \beta < 1$, $\theta = (A_1, B_1, \omega_1, \cdots, A_q, B_q, \omega_q)$ and φ_β is a check function which is defined by the following formula.

$$\varphi_\beta(\lambda) = \begin{cases} \beta\lambda, & \lambda \geq 0, \\ (\beta - 1)\lambda, & \lambda < 0. \end{cases}$$

On the other hand, Oberhofer(1982) studied the weak consistency about the least absolute deviation(LAD) estimators - the special case of the RQE for $\beta = 0.5$, with the assumptions from B1 to B6 in his paper, but the assumption B5 is equivalent to assumption of Jennrich(1969). So in order to prove the asymptotic properties of the RQE for this model we must take the different method. First of all the Section 2 and 3 provide the asymptotic properties of the RQE for one harmonic component case, the case of several harmonic components is given in section 4.

2. The strong consistency

Here, for the case of $q = 1$, we will consider the strong consistency of the nonlinear RQE $\hat{\theta}_T = \hat{\theta}_{1T} = (\hat{A}_{1T}, \hat{B}_{1T}, \hat{\omega}_{1T}) = (\hat{A}_T, \hat{B}_T, \hat{\omega}_T)$ for $\theta_o = (A_{1o}, B_{1o}, \omega_{1o}) = (A_o, B_o, \omega_o)$ in a time series with stationary independent residuals model (1.1) with the following assumptions.

Assumption A

The parameter space $\Theta = K \times K \times [0, \pi]$, where K is compact subspace of R.

Assumption B

B1: ϵ_t are i.i.d. random variables with the common distribution function G and continuous probability density function $g(x)$ such that $G(0) = \beta$, where $0 < \beta < 1$, and $g(0) > 0$.

B2: $E\{{\epsilon_t}^2\} < \infty$, for all t.

Theorem 2.1 *Suppose that Assumption A and B are satisfied on the model (1.1). Then the RQE $\hat{\theta}_T$ is strongly consistent for θ_o.*

3. Asymptotic normality

Next we consider the asymptotic normality of the proposed estimator $\hat{\theta}_T$ which is one of the most important statistical properties in asymptotic theory.

Theorem 3.1 *For the given model (1.1) with the assumptions A and B, we conclude that* $\left(\sqrt{T}(\hat{A}_T - A_0), \sqrt{T}(\hat{B}_T - B_0), \sqrt{T}^3(\hat{\omega}_T - \omega_0)\right)$ *converges in law* $N\left((0,0,0), \frac{\beta(1-\beta)}{g^2(0)}\Sigma^{-1}\right)$, where

$$\Sigma = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4}B_0 \\ 0 & \frac{1}{2} & -\frac{1}{4}A_0 \\ \frac{1}{4}B_0 & -\frac{1}{4}A_0 & \frac{(A_0^2+B_0^2)}{6} \end{pmatrix}.$$

4. The case of several harmonic components

Suppose now that the model in (1.1) is generalized to $q > 1$. The function corresponding to (1.2) whose minimization yields estimators $\hat{\theta}_T = (\hat{A}_{1T}, \hat{B}_{1T}, \hat{\omega}_{1T}, \cdots, \hat{A}_{qT}, \hat{B}_{qT}, \hat{\omega}_{qT})$ becomes (1.2), where $\theta = (A_1, B_1, \omega_1, \cdots, A_q, B_q, \omega_q)$. With the following same condition in the Walker(1971),

$$\lim_{T\to\infty} \min_{1\le r\ne s\le q} \left(T|\omega_r - \omega_s|\right) = \infty, \tag{4.1}$$

Using assumption A, B and the above condition, we can show the strong consistency likewise Theorem 2.1.

Also likewise the theorem 3.1, we obtain the following theorem.

Theorem 4.1 *With the same conditions of Theorem 3.1 and (4.1),* $P(\hat{\theta}_T) = (P_1(\hat{\theta}_{1T}), P_2(\hat{\theta}_{2T}), \cdots, P_q(\hat{\theta}_{qT}))$, *where* $P_r(\hat{\theta}_{rT}) = \left(\sqrt{T}(\hat{A}_{rT} - A_{ro}), \sqrt{T}(\hat{B}_{rT} - B_{ro}), \sqrt{T}^3(\hat{\omega}_{rT} - \omega_{ro})\right)(1 \le r \le q)$ *converges in law* $N\left(\mathbf{0}_{3q\times 1}, \frac{\beta(1-\beta)}{g^2(0)}\Sigma^{-1}\right)$, *where* $\Sigma = \left(\Sigma_{rs}\right)_{3q\times 3q}$, *for* $r, s = 1, 2, \cdots, q$,

$$\Sigma_{rs} = \begin{cases} \mathbf{0} & \text{if } r \ne s, \\ \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4}B_{ro} \\ 0 & \frac{1}{2} & -\frac{1}{4}A_{ro} \\ \frac{1}{4}B_{ro} & -\frac{1}{4}A_{ro} & \frac{(A_{ro}^2+B_{ro}^2)}{6} \end{pmatrix} & \text{if } r = s. \end{cases}$$

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