Proceedings of the 14th
KACC, Octorber 1999

Trajectory Optimization Operations for Satellites in Elliptic Orbits

Chang-Hee Won, Hee Sook Mo, In Jun Kim and Seong-Pal Lee
Satellite Communications System Department,
Electronics and Telecommunications Research Institute,
Yusong P.O. Box 106, Taejon, Korea 305-600

Abstract

Minimum-fuel and -time orbit transfer are two major goals of the satellite trajectory optimization. In this paper, we consider satellites in two coplanar elliptic orbits when the apsidal lines coincide, and analytically find the conditions for the two-impulse minimum-time transfer orbit using Lambert's theorem. The transfer time is a decreasing function of a variable related to the transfer orbit's semimajor axis in the minimum-time case. In the minimum-time case, there is no unique minimum-time solution, but there is a limiting solution. However, there exists a unique solution in the case of minimum-fuel transfer, for which we find analytically the necessary and sufficient conditions. As a special case, we consider when the transfer angle is one hundred and eighty degrees. In this case, we show that we obtain the classical fuel-optimal Hohmann transfer orbit. We also derive the Hohmann transfer time and delta-velocity equations from more general equations, which are obtained using Lambert's theorem. We note the tradeoff between minimum-time and -fuel transfer. An optimal coplanar orbit maneuver algorithm to trade off the minimum-time goal against the minimum-fuel goal is proposed. Finally, the numerical simulation results are given to demonstrate the derived theory and the algorithm.

1. Introduction

Trajectory optimization to minimize the fuel usage was done by Lawden[1] using the primer vector in 1963, however the optimal-time transfer was not addressed. The tradeoff between time and fuel for the rendezvous problem was addressed by Prussing and Chiu[2], who came up with the minimum-fuel, multiple-impulse, time fixed solutions for coplanar circular rendezvous problem. They also proved that the Hohmann transfer is the time-open solution for the optimal rendezvous. They developed an iterative minimization method to find the optimal number of impulses and their positions and time using Lawden's primer vector. It is well-known that the Hohmann transfer is the minimum-fuel two-impulse transfer between two coplanar circular orbits[3]. For trajectory optimization problem, see Bryson and Ho.[4]. And see Betts[5] for a general survey of numerical methods for trajectory optimization.

Lambert's theorem is typically applied to the transfer orbit determination from the connecting two position vectors and the transfer time. See[6,7] and the references cited within. It is possible to perform, however, the optimal-fuel or -time transfer using Lambert's theorem. In this paper we formulate the transfer time and delta-velocity equations utilizing Lambert's theorem. Then we derive the minimum-time transfer condition for the two-impulse elliptic-to-elliptic transfer. The necessary and sufficient conditions are also derived for minimum-fuel transfer when the transfer angle is anywhere between 0~360 degrees. Furthermore, it is shown that the Hohmann transfer is the minimum-fuel two-impulse solution when the transfer angle is 180 degrees.

This optimal transfer is being developed with satellite orbit maneuvers, especially satellite altitude maneuvers of a low earth orbiting satellite, in mind. We have developed a mission analysis and planning (MAPS)software package for Korea Multipurpose satellite. In the next MAPS software we may implement this idea and send the optimizing commands to Satellite Operations Subsystem (SOS) in Fig. 1.

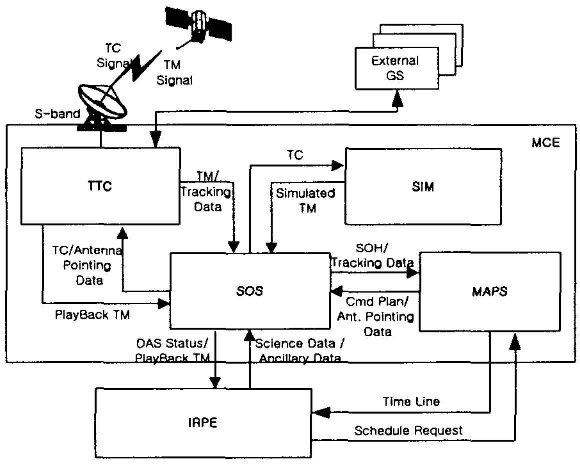

Fig. 1 Ground Station System Structure

On the other hand, Another major factor in our development is to optimize the transfer time. In this case, we seek the minimum-time solution assuming that two-impulses are utilized. This can be applied to the spacecraft interception and rendezvous problems. We present the theories and algorithms involved in the fuel versus time tradeoff.

In this paper, we find the two-impulse minimum-time transfer orbit analytically. Moreover, we extract the necessary and sufficient conditions for the minimum-fuel orbit transfer problem using Lambert's theorem. This paper also highlights an algorithm to find the tradeoff between minimum-fuel and -time orbit transfer.

Problem Formulations

This paper assumes two elliptic coplanar orbits when their apsidal lines coincide. Fig. 2 shows the initial orbit radius r_1, target orbit radius r_2, the initial orbit semimajor axis a_1, the target orbit semimajor axis a_2, the difference in the true anomaly (transfer angle) Δf, and the cord length, c. The semi-perimeter of the space triangle is given by

$$s = \frac{r_1 + r_2 + c}{2}. \quad (1)$$

Following Battin's formulation[8], define a constant λ by

$$\lambda^2 = \frac{s-c}{s}. \quad (2)$$

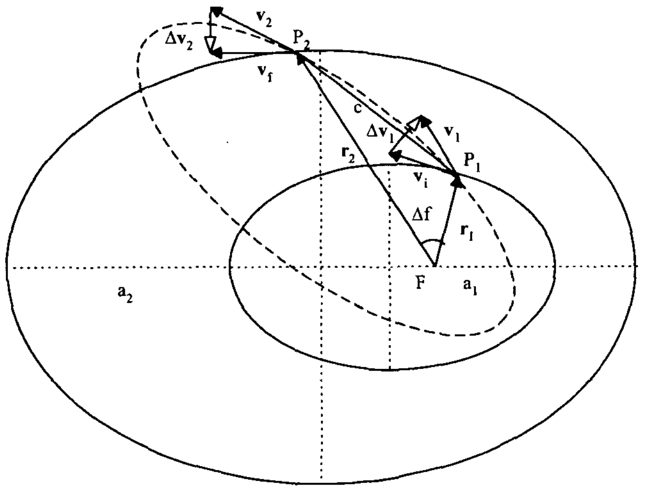

Fig. 2 Coplanar orbit transfer diagram.

Note that $\lambda \in [-1,1]$ and it is closely related to the transfer angle, Δf. Thus, λ relates to the cord length. For example, $\lambda = 0$ implies $\Delta f = \pi$ and $\lambda = \pm 1$ implies $\Delta f = 0$.

Also, the variable x is defined by

$$x^2 = 1 - \frac{a_m}{a}, \quad (3)$$

where a_m is the semi-major axis of the minimum-energy elliptic transfer orbit given as

$$a_m = \frac{s}{2}, \quad (4)$$

and a is the semi-major axis of the transfer orbit.

Transfer Time Equation

Using Lambert's theorem, we have the transfer time equation for elliptic (hyperbolic) transfer orbits as[8]

$$\Delta t = \sqrt{\frac{a_m^3}{\mu}} \left(\frac{\alpha - \sin\alpha}{\sin^3 \frac{\alpha}{2}} - \lambda^3 \frac{\beta - \sin\beta}{\sin^3 \frac{\beta}{2}} \right) \quad (5)$$

where $\mu = 3.986 \times 10^5$ km^3/s^2 is the gravitational constant of the Earth. For hyperbolic orbits, we replace "sin" with "sinh" and multiply Eq. (5) by -1. Following Battin[8] we define

$$\alpha = 2\arccos(x), \quad (\text{hyperbolic; } x = \cosh\alpha/2) \quad (6)$$

$$\beta = 2\arccos(y), (\text{hyperbolic; } y = \cosh\beta/2) \text{ and} \quad (7)$$

$$y = \sqrt{1 - \lambda^2(1 - x^2)}. \quad (8)$$

Thus we have expressed the transfer time equation in terms of the new variable x. Eq. (5) can be utilized to find the minimum-time transfer orbit. In the next section, we differentiate Δt with respect to x to find the minimum transfer time orbit.

Before we consider this derivative, however, it is helpful to express α and β in terms of geometric parameters, s, c, and a. When Eqs. (4) is substituted into Eq. (3), we obtain

$$x = \pm\sqrt{1 - \frac{s}{2a}}. \quad (9)$$

Equating the above equation with Eq. (6) and using a simple trigonometric identity we obtain

$$\sin^2 \frac{\alpha}{2} = \frac{s}{2a}. \quad (10)$$

Similarly, substitute Eq. (2) and (9) into Eq. (8) to obtain $y = \sqrt{1-(s-c)/(2a)}$. Also from the definition (7) and a simple trigonometric identity we obtain

$$\sin^2 \frac{\beta}{2} = \frac{s-c}{2a}. \quad (11)$$

Finally, we have α and β in terms of s, c, and a in Eqs. (10) and (11)

Delta Velocity Equation

We can find the relation between the transfer orbit's semimajor axis and the fuel usage from the total delta-velocity vector, utilizing Lambert's theorem[8]. In this section, the total delta-velocity vector shall be written as a function of the variable x, to perform fuel optimization in the sequel.

Let $\mathbf{v}_1$ be the transfer orbit velocity at P_1 and $\mathbf{v}_i$ be the initial orbit velocity at P_1 (Fig. 1). Then the delta-velocity is defined as

$$\Delta\mathbf{v}_1 = \mathbf{v}_1 - \mathbf{v}_i. \quad (12)$$

$$\mathbf{v}_1 = \frac{1}{\eta}\sqrt{\frac{\mu}{a_m}} \left\{ \left[2\lambda\frac{a_m}{r_1} - (\lambda + x\eta) \right] \mathbf{i}_{r1} + \left[\sqrt{\frac{r_2}{r_1}} \sin\frac{\Delta f}{2} \right] \mathbf{i}_n \times \mathbf{i}_{r1} \right\}, \quad (13)$$

where $\mathbf{i}_{r1}$ is a unit vector defining the direction of P_1 from the force center and $\mathbf{i}_n$ is a unit vector normal to the orbital plane. Define η by

$$\eta^2 = (1 - \lambda^2) + 4\lambda \sin^2 \frac{\psi}{2} \quad (14)$$

for an elliptic orbit. Moreover, ψ is given as

$$\psi = \frac{\alpha - \beta}{2} \quad (15)$$

where α and β are given in (10) and (11). Also, because we assumed two elliptic coplanar orbits, the initial orbit velocity is given by

$$\mathbf{v}_i = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_1}\right)}\mathbf{i}_n \times \mathbf{i}_{r1} . \quad (16)$$

Similarly, let $\mathbf{v}_2$ be the transfer orbit velocity at P_2 and $\mathbf{v}_f$ be the target orbit velocity at P_2:

$$\Delta\mathbf{v}_2 = \mathbf{v}_f - \mathbf{v}_2 , \quad (17)$$

$$\mathbf{v}_2 = \frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left\{\left[2\lambda\frac{a_m}{r_2} - (\lambda + x\eta)\right]\mathbf{i}_{r2} + \left[\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2}\right]\mathbf{i}_n \times \mathbf{i}_{r2}\right\}$$

and (18)

$$\mathbf{v}_f = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_2}\right)}\mathbf{i}_n \times \mathbf{i}_{r2} . \quad (19)$$

Then the total $\Delta\mathbf{v}_{total}$ is given as the sum of Eqs. (12) and (17) as

$$\Delta\mathbf{v}_{total} = \Delta\mathbf{v}_1 + \Delta\mathbf{v}_2 \quad (20)$$

where

$$(\Delta v_1)^2 = \left\{\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left[2\lambda\frac{a_m}{r_1} - (\lambda + x\eta)\right]\right\}^2 + \left[\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_2}{r_1}}\sin\frac{\Delta f}{2} - \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_1}\right)}\right]^2 \text{ and } (21)$$

$$(\Delta v_2)^2 = \left\{-\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left[2\lambda\frac{a_m}{r_2} - (\lambda + x\eta)\right]\right\}^2 + \left[-\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2} + \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_2}\right)}\right]^2 \quad (22)$$

To minimize the fuel usage of a spacecraft, $\Delta\mathbf{v}_{total}$ must be minimized. Consequently, we minimize Eq. (20) with respect to x and find the minimum-fuel transfer.

Note that non-coplanar orbits may be considered in this manner to find the optimal-fuel and -time elliptic transfer orbit. In the non-coplanar case, Eq. (5) would remain the same, however, $\mathbf{i}_n$ in Eqs. (13) and (18) would change to $\mathbf{i}_{nt}$, $\mathbf{i}_n$ in Eq. (16) would change to $\mathbf{i}_{n1}$, and $\mathbf{i}_n$ in Eq. (19) would change to $\mathbf{i}_{n2}$. Here $\mathbf{i}_{nt}$ is a unit vector normal to the transfer orbital plane, $\mathbf{i}_{n1}$ is a unit vector normal to the initial orbital plane, and $\mathbf{i}_{n2}$ is a unit vector normal to the target orbital plane.

Minimum-Time Transfer Conditions

Derivation of the Transfer Time Equation

The conditions to achieve two-impulse optimal-time transfer are derived. We show that the transfer time is a decreasing function of x. Thus minimum-time transfer does not have a definite solution, but it has a limiting solution in the sense that it is possible to find the limiting x such that the minimum transfer time approaches this limit. The results are stated in this section and the proofs are given in the reference[9].

Taking derivative of Eq. (5) with respect to x, we can prove[9] that the transfer time equation is a monotonically decreasing function with respect to the variable, x, i.e., $d\Delta t / dx < 0$. Consequently the largest value of x gives the fastest transfer time. For example, to find the fastest elliptic transfer orbit, we let x approach unity without actually reaching it ($x = 1$ corresponds to a parabolic orbit). Furthermore, this implies that the parabolic transfer orbits are faster than elliptic transfer orbits, and hyperbolic transfer orbits are faster than parabolic transfer orbits. Intuitively this states that if fuel is unlimited, the more delta-velocity one expends the shorter the transfer time.

Necessary and Sufficient Conditions for Minimum-Fuel Transfer

We extract the necessary and sufficient conditions for the minimum-fuel transfer in the case of a two-impulse elliptic transfer orbit in this section. We assume that the initial and target orbits is coplanar and coaxial elliptic orbits. We find the condition on the variable x (equivalent to the semimajor axis) such that the transfer orbit minimizes the fuel usage. The magnitudes of the delta velocities (Δv_1, and Δv_2) are given by the following Eqs. (21) and (22). By definition, η is positive[8] and it can be rewritten as

$$\eta = y - \lambda x = \sqrt{1 - \lambda^2(1 - x^2)} - \lambda x \quad (23)$$

Given Δf, the necessary and sufficient conditions for a minimum-fuel transfer orbit are given by $\frac{d\Delta v_{total}}{dx} = 0$ and $\frac{d^2\Delta v_{total}}{dx^2} > 0$, where $\frac{d\Delta v_{total}}{dx} = \frac{d\Delta v_1}{dx} + \frac{d\Delta v_2}{dx}$.

Necessary Condition

Thus, to find the necessary and sufficient conditions, we take the first derivative of the delta-velocities given in Eqs. (21) and (22) with respect to x. Also, we find the first derivative of Eq. (29) with respect to x. We obtain

$$
\begin{aligned}
\frac{d\Delta v_1}{dx} = {} & \frac{1}{\Delta v_1}\frac{\mu}{a_m}\left[\frac{1}{\eta}\left(2\lambda\frac{a_m}{r_1}-(\lambda+x\eta)\right)\right] \\
& \left(-\frac{1}{\eta^2}2\lambda\frac{a_m}{r_1}\frac{d\eta}{dx}+\frac{1}{\eta^2}\lambda\frac{d\eta}{dx}-1\right) \\
& +\frac{1}{\Delta v_1}\left(\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_2}{r_1}}\sin\frac{\Delta f}{2}-\sqrt{\mu\left(\frac{2}{r_1}-\frac{1}{a_1}\right)}\right) \\
& \left(-\frac{1}{\eta^2}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_2}{r_1}}\sin\frac{\Delta f}{2}\frac{d\eta}{dx}\right),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{d\Delta v_2}{dx} = {} & \frac{1}{\Delta v_2}\frac{\mu}{a_m}\left[-\frac{1}{\eta}\left(2\lambda\frac{a_m}{r_2}-(\lambda+x\eta)\right)\right] \\
& \left(\frac{1}{\eta^2}2\lambda\frac{a_m}{r_2}\frac{d\eta}{dx}-\frac{1}{\eta^2}\lambda\frac{d\eta}{dx}+1\right) \\
& +\frac{1}{\Delta v_2}\left(-\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2}+\sqrt{\mu\left(\frac{2}{r_2}-\frac{1}{a_2}\right)}\right) \\
& \left(\frac{1}{\eta^2}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2}\frac{d\eta}{dx}\right), \text{ and}
\end{aligned}
\tag{25}
$$

$$\frac{d\eta}{dx}=\lambda^2\frac{x}{y}-\lambda. \tag{26}$$

The first derivative of the total delta velocity,

$$\frac{d\Delta v_{total}}{dx}=\frac{d\Delta v_1}{dx}+\frac{d\Delta v_2}{dx}, \tag{27}$$

can be trivially obtained from Eqs. (24) and (25). Thus the necessary condition for a minimum requires $\frac{d\Delta v_{total}}{dx}=0$.

Sufficient Condition

For the sufficient condition, we find the second derivative of Eq. (26) with respect to x.

$$\frac{d^2\eta}{dx^2}=\lambda^2\frac{y-x\frac{dy}{dx}}{y^2} \tag{28}$$

Then we take the second derivative of the total delta velocity with respect to x. This second derivative must be greater than zero for Δv_{total} to be minimum. Thus we have the sufficient condition for a local minimum;

$$\frac{d^2\Delta v_{total}}{dx^2}>0. \tag{29}$$

Equating (27) to zero gives the necessary condition and (29) gives the sufficient condition to achieve minimum-fuel transfer for any two points in space (i.e., Δf can be anywhere between zero and 360 degrees).

Coplanar Hohmann Transfer From Lambert's Theorem

Necessary and Sufficient Conditions for a Minimum

To verify the results of the previous section, we assume elliptic coplanar coaxial orbits and derive the necessary and sufficient conditions for the Hohmann transfer case. By Hohmann transfer, we mean that tangential impulses are applied at opposing apses. Consequently, the initial and target points must be on the line of apsides for Hohamnn transfer. We show that for the Hohmann transfer x is equal to zero. In the Hohmann transfer, we have $\Delta f=\pi$ and $c=r_1+r_2$. Thus from Eqs. (1) and (2), we get $\lambda=0$. Moreover, because $\lambda=0$, from Eq. (14) we obtain $\eta=1$ (η is positive by definition). Substituting these values into Eq. (27) gives the following result:

$$\frac{d\Delta v_{total}}{dx}=\frac{\mu}{a_m}x\left(\frac{1}{\Delta v_1}+\frac{1}{\Delta v_2}\right). \tag{30}$$

Thus $d\Delta v_{total}/dx$ is equal to zero when x is equal to zero. Consequently, for the case of a Hohmann transfer ($\Delta f=\pi$), we obtain the optimal fuel transfer when x is equal to zero. This shows that condition (30) is a special case of the general condition (27). Similarly the sufficient condition can be shown.[9]

Note that $x=0$ corresponds to the minimum energy transfer[8], and for Hohmann transfer the minimum energy transfer corresponds to the minimum-fuel transfer, but as will be seen in the simulation section, when $\Delta f\neq\pi$ this is not the case.

Derivation of Transfer Time and Delta Velocity Equations

From the Lagrange form of the transfer time Eq. (5), we obtain the Hohmann transfer time equation,

$$\Delta t=\pi\sqrt{\frac{(r_1+r_2)^3}{8\mu}} \tag{31}$$

when Δf is equal to 180 degrees. We note that this transfer time equation is valid for elliptic-to-elliptic coaxial orbits as well as the circle-to-circle orbits.

We can also recover the well-known Hohmann total delta-velocity, $\Delta \mathbf{v}_{total}$,

$$\Delta \mathbf{v}_{total}=\sqrt{\mu}\left[\left|\left(\frac{2}{r_1}-\frac{1}{a}\right)^{\frac{1}{2}}-\left(\frac{1}{r_1}\right)^{\frac{1}{2}}\right|+\left|\left(\frac{2}{r_2}-\frac{1}{a}\right)^{\frac{1}{2}}-\left(\frac{1}{r_2}\right)^{\frac{1}{2}}\right|\right] \tag{32}$$

when Δf is equal to 180 degrees.

Note that the Hohmann transfer is a special case for Δf equal to 180 degrees. Thus for $\Delta f\neq 180$ degrees, the Hohmann transfer cannot be used, however, we can still obtain the minimum-fuel or minimum-time transfer orbit using the method described in the previous section.

Optimal Coplanar Orbit Maneuver Algorithm

Here, a procedure to satisfy the optimal-fuel and -time maneuver using Lambert's theorem is described. The problem is to find the transfer orbit's semimajor axis such that the transfer time and the fuel used is minimum. The optimal algorithm is as follows:

1. Given initial orbit radius, r_1, target orbit radius, r_2, and the difference in the true anomaly (transfer angle), Δf, we vary x to find the transfer time, Δt, and total delta velocity, $\Delta \mathbf{v}_{total}$.
2. Find the cord length, c, from the equation,

$$c^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\Delta f \,. \tag{33}$$

3. Find the semiperimeter of the space triangle, s, from Eq. (1).
4. Find a_m from Eq. (4).
5. Find a constant λ from Eq. (2).

Note that the cord length, c, can also be expressed in terms of this variable as

$$c = (r_1 + r_2)\frac{1-\lambda^2}{1+\lambda^2} \,. \tag{42}$$

Thus λ is related to the cord length.

6. Find α from Eq. (6).
7. Find β from Eq. (7).
8. Find the transfer time from the Lagrange form of the transfer time Eq. (5).
9. Find the total delta velocity from Eqs. (20), (21), and (22).

Eq. (5) can be used to find the transfer orbit's semimajor axis such that the transfer time is minimized. We can plot Δt (which is related to the transfer orbit semi-major axis) versus x, and find x such that Δt is minimum. Actually it is known from the previous section that Δt is a decreasing function of x. Thus for example, for an elliptic transfer orbit the smallest Δt is achieved by letting x approach unity without actually reaching it ($x = 1$ corresponds to a parabolic orbit). Thus, just to perform the elliptic minimum-time orbit transfer we let x be very close to unity (for elliptic transfer orbit $x < 1$).

There is another factor to take into consideration when we perform an optimal maneuver. We would like to minimize fuel usage while transferring to the target orbit. We can plot $\Delta \boldsymbol{v}_{total}$ (which is related to the transfer orbit semi-major axis) versus x using Eq. (20), and find x such that $\Delta \boldsymbol{v}_{total}$ is minimum to find the optimal-fuel transfer orbit. Thus there is a tradeoff between the time the spacecraft takes to get to the target orbit and the fuel necessary to get to the target orbit.

Numerical Simulation Results

In this section, the numerical simulation is performed to verify the optimal maneuver algorithm developed in the previous section. Here we let $r_1 = 6700$ km, $r_2 = 6710$ km, $a_1 = 6800$ km, and $a_2 = 6900$ km. The first example is when $\Delta f = \pi$ (Hohmann transfer) and the second example is when $\Delta f = \pi/10$.

Case 1 ($\Delta f = \pi$: Hohmann Transfer)

We plot $|\Delta \mathbf{v}_{total}|$ versus x for a Hohmann transfer in Fig. 3. Note that minimum $|\Delta \mathbf{v}_{total}|$, which has the value 0.1619 km/s, corresponds to $x = 0$, which represents the Hohmann transfer. Using Eqs. (27) and (29), we have verified that $d\Delta v_{total}/dx = 0$ and $d^2\Delta v_{total}/dx^2 > 0$ when $x = 0$. Thus $x = 0$ corresponds to the minimum-fuel solution.

Fig. 4 shows transfer time versus x. Note that it is a monotonically decreasing function. When x is equal to zero,

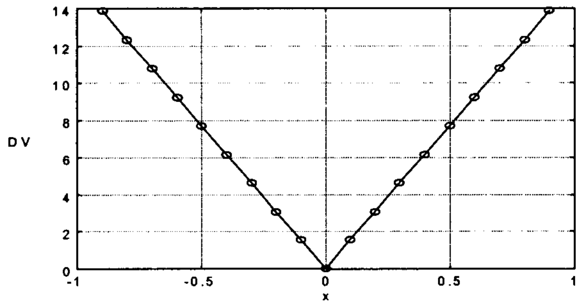

Fig. 3. $|\Delta \boldsymbol{v}_{total}|$ versus x ($\Delta f = \pi$).

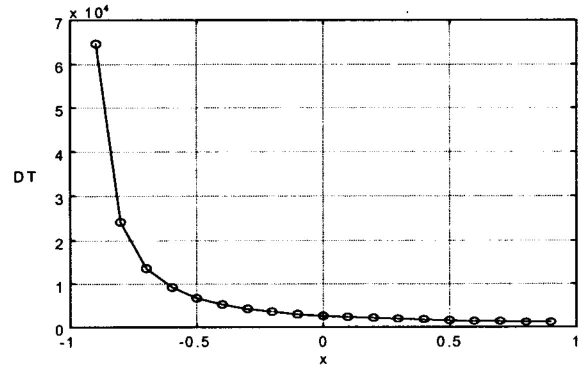

Fig. 4. Transfer time versus x ($\Delta f = \pi$).

the transfer time is 2732.0 seconds. It is possible to obtain a transfer orbit that will give a smaller transfer time, but the fuel used will increase. Thus there is tradeoff between the fuel used and minimum-time. Although it is not shown on the plot, the parabolic transfer orbit has smaller transfer time than an elliptic transfer orbit, and the hyperbolic transfer orbit has smaller transfer time than a parabolic transfer orbit.

Case 2 ($\Delta f = \pi/10$)

In the case of rendezvous, we may require that $\Delta f \neq \pi$. Thus the question is what is the optimal-fuel or -time transfer orbit when $\Delta f \neq \pi$. Here we assume that the two points are separated by $\Delta f = \pi/10$. Fig. 5 shows $|\Delta \mathbf{v}_{total}|$ versus x. Note that the minimum $|\Delta \mathbf{v}_{total}|$ is achieved when x is approximately 0.65. Note also that the minimum energy orbit ($x = 0$) does not correspond to the minimum-fuel orbit. Using Eqs. (33) and (35), we have verified that $d\Delta v_{total}/dx = 0$ and $d^2\Delta v_{total}/dx^2 > 0$ when $x \approx 0.649$. Thus $x \approx 0.649$ corresponds to the minimum-fuel solution. See the $d\Delta v_{total}/dx$ versus x graph in Fig. 6.

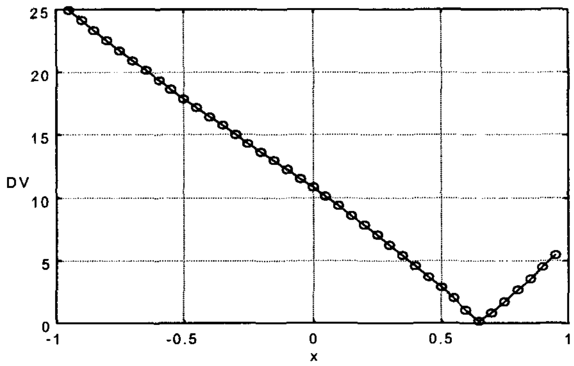

Fig.5. $|\Delta v_{total}|$ versus x ($\Delta f = \pi/10$).

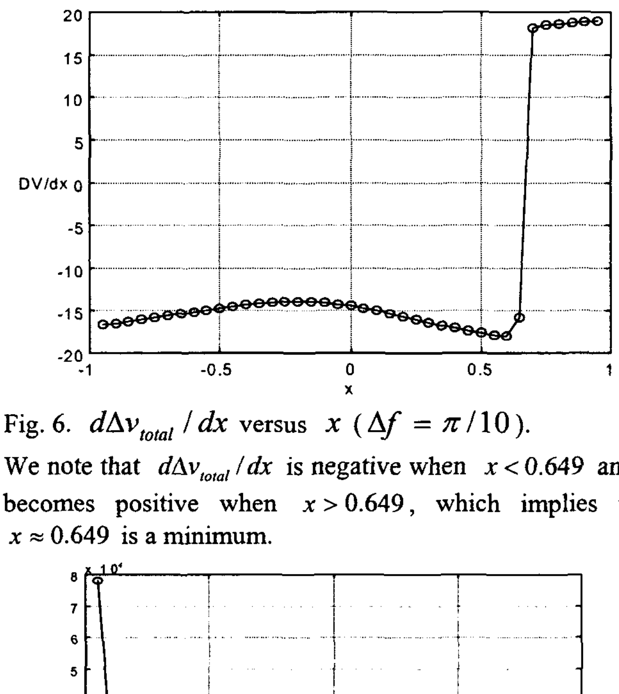

Fig. 6. $d\Delta v_{total}/dx$ versus x ($\Delta f = \pi/10$).

We note that $d\Delta v_{total}/dx$ is negative when $x < 0.649$ and it becomes positive when $x > 0.649$, which implies that $x \approx 0.649$ is a minimum.

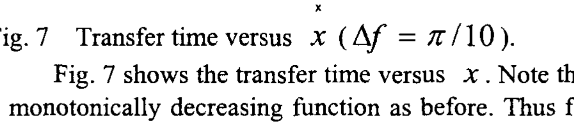

Fig. 7 Transfer time versus x ($\Delta f = \pi/10$).

Fig. 7 shows the transfer time versus x. Note that it is a monotonically decreasing function as before. Thus for this problem the optimal fuel and time transfer orbit may be when $x = 0.65$, which corresponds to $a = 6.7133 \times 10^3$ km. For $x = 0.65$, $\Delta v_{total} = 0.1703$ km/s and $\Delta t = 273.03$ sec.

To compare these values with the Hohmann transfer case, see Table 1, case 2 is for $\Delta f = \pi/10$. Note that Δv_{total} is smaller for the Hohmann transfer (Case 1, $\Delta f = \pi$) as expected but the transfer time is much larger.

Table 1 Comparison of Case 1 and Case 2

	x	Δv_{total} (km/s)	Δt (second)	a (km)
Case 1	0.00	0.1619	2732.0	6705.0
Case 2	0.65	0.1703	273.03	6713.3
Diff.	0.65	0.0084	-2459.0	8.3000

Conclusions

The transfer time and delta-velocity equations have been described using Lambert's theorem. Utilizing these equations, minimum-time and minimum-fuel conditions in terms of the semimajor axis of the elliptic transfer orbit can be derived. Moreover, it is shown that the minimum-time transfer orbit approaches a limiting value. It is also shown that when the transfer angle is 180 degrees the optimal fuel transfer is equivalent to the well-known Hohmann transfer. An algorithm with the fuel and time tradeoff is presented. This procedure can be used to find the optimal-time and -fuel transfer orbit in the case of a rendezvous.

References

[1] Lawden, D.F., *Optimal Trajectories for Space Navigation*, Butterworths & Co., London, 1963, pp. 96-124.

[2] Prussing, John E., and Chiu, Jeng-Hua, "Optimal Multiple-Impulse Time-Fixed Rendezvous Between Circular Orbits," *Journal of Guidance, Control, and Dynamics*, Vol. 9, No. 1, January-February 1986, pp. 17-22.

[3] Prussing, John E., "Simple Proof of the Global Optimality of the Hohmann Transfer," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 4, July-August 1992, pp. 1037-1038.

[4] Bryson, Arthur E., and Ho, Yu-Chi, *Applied Optimal Control: Optimization, Estimation, and Control*, Hemisphere Publishing Co., New York, 1975, pp. 42-89.

[5] Betts, John T., "Survey of Numerical Methods for Trajectory Optimization," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 2, March-April 1998.

[6] Battin, Richard H., and Vaughan, Robin M., "An Elegant Lambert Algorithm," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 6, November-December 1984, pp. 662-670.

[7] Palmore, Julian, "An Elementary Proof of the Optimality of Hohmann Transfers," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 5, September-October 1984.

[8] Battin, Richard H., *An Introduction to the Mathematics and Methods of Astrodynamics*, edited by J. S. Przemieniecki, AIAA Education Series, New York, 1987, pp. 289-307.

[9] Chang-Hee Won, "Fuel- or Time-Optimal Transfers Between Coplanar, Coaxial Ellipses Using Lambert's Theorem," *Journal of Guidance, Control, and Dynamics*, Vol.22, No. 4, July-August 1999, pp. 536-542.