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Stability Analysis of Fuzzy-Model-Based Controller by Piecewise Quadratic Lyapunov Functions

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Abstract

In this paper, piecewise quadratic Lyapunov functions are used to analyze the stability of fuzzy-model-based controller. We represent the nonlinear system using a Takagi-Sugeno fuzzy model, which represent the given nonlinear system by fuzzy inference rules and local linear dynamic models. The proposed stability analysis technique is developed by dividing the whole fuzzy system into the smaller separate fuzzy systems to reduce the conservatism. Some necessary and sufficient conditions for the proposed method are obtained. Finally, stability of the closed system with various kinds of controller for TS fuzzy model is checked through the proposed method.

1. Introduction

During the past several years, fuzzy logic control has become one of well-received approaches for utilizing qualitative knowledge of a plant to design a controller [1-5, 8-10]. Fuzzy logic control is generally applicable to the plants that are mathematically poorly modeled and where the qualitative knowledge of experienced operators is available for qualitative control. The specific design of a fuzzy logic controller, however, has difficulties in the acquisition of expert's knowledge and relies to a great extent on empirical and heuristic knowledge that, in many cases, cannot be justified.

In 1985, Takagi and Sugeno proposed a new kind of fuzzy inference system, called the Takagi–Sugeno (TS) fuzzy model. It can combine the flexibility of fuzzy logic theory and the rigorous mathematical analysis tools into a unified framework. Since it employs linear models in the consequent parts, it is convenient to apply the conventional linear systems theory for analysis. Since then, various kinds of TS fuzzy model based controllers have been suggested [1-5]. In these methods, sets of fuzzy rules are used to construct suitable local linear state models from which local controllers can be determined. The stability of the overall system is then determined by a Lyapunov stability analysis and LMI method. On the other hand, Cao et al. [3-5] developed another kind of fuzzy controller structure. In their works, the switching type controller is used and the stability condition is derived by solving some Algebraic Riccati Equations

Approach in [1-2] suffers mainly from a point that a common positive definite matrix must be found to satisfy several Lyapunov equations, which can be difficult especially when the number of fuzzy rules required to give a good plant model is large. Although LMI based approaches have been used to determine the existence of a common positive definite matrix [1-2], some other problems still remain unresolved. Many researchers are concentrated on relaxing the stability condition of fuzzy-model-based controller.

Another kind of approach proposed by Cao et al. [3-5] They use uncertain linear system theory to analyze the stability of the fuzzy-model-based controller. It also shows conservatism since the upper bounds which represent the interactions between fuzzy rules can not be exactly determined.

The purpose of this work is construct more relaxed stability condition of fuzzy-model-based controller with the techniques in modern control theory.

The main contribution of this work can be summarized as follows. It is shown that the TS fuzzy model can be reduced to the smaller TS fuzzy models if the number of fuzzy rules fired at an instant is smaller than the number of rules in the fuzzy rule base. The stability analysis of the fuzzy-model-based controller is performed with piecewise quadratic (PQ) Lyapunov function.

This paper is organized as follows: The preliminaries are first briefly reviewed in Section 2. In Section 3, the proposed stability analysis technique is detailed. Finally, conclusions are drawn in Section 4.

2. Preliminaries

In this paper, all matrices are with appropriate dimensions if the dimensions are not explicitly stated.

The nonlinear systems can be modeled by finite dimensional, deterministic ordinary differential equations as follows:

$$\dot{x} = f(x(t), u(t), t) \tag{1}$$

in which $x \in \Re^n$ is the state, $u(t): \Re^+ \to \Re^p$ is the control input. However, exact determination of the nonlinear dynamic equations for the given nonlinear system in the form of (1) is very difficult since the real physical system is very complex. Even if one can obtain the exact nonlinear dynamic equations, the design of the suitable controller for the nonlinear system is complicated. In many cases, the nonlinear system (1) may be approximated by multiple linear state models. TS fuzzy model combines the fuzzy inference rule and the local linear state models [1-2]. The *i*th rule of the TS fuzzy model, representing the complex system (1), is the following:

Plant Rule *i*:

$$\begin{aligned} &\text{IF } x_1(t) \text{ is } F_1^i \text{ and ... and } x_j(t) \text{ is } F_n^i \\ &\text{THEN } \dot{x}(t) = A_i x(t) + B_i u(t) \quad (i = 1, 2, \ldots, r) \end{aligned} \tag{2}$$

where Rule *i* denotes the *i*th fuzzy inference rule, F_j^i (j = 1, 2, ..., n) are fuzzy sets, $x \in \Re^n$ is the state vector, $u(t): \Re^+ \to \Re^p$ is the input control, $A_i \in \Re^{n \times n}$, $B_i \in \Re^{n \times p}$, r is the number of fuzzy IF-THEN rules, and the constant matrices

By using the fuzzy inference method with a singleton fuzzifier, product inference, and center average defuzzifier, the dynamic fuzzy model (2) can be expressed as the following global model:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)) \\ &= A(\mu(x(t)))x(t) + B(\mu(x(t)))u(t) \end{aligned} \tag{3}$$

where,

$$w_i(x(t)) = \prod_{j=1}^{n} F_j^i(x^{(j-1)}(t))$$

$$\mu_i(x(t)) = \frac{w_i(x(t))}{\sum_{i=1}^{r} w_i(x(t))}$$

$$\mu(x(t)) = (\mu_1(x(t)), \mu_2(x(t)), \cdots, \mu_n(x(t)))$$

and $F_j^i(x_j(t))$ is the grade of membership of $x_j(t)$ in F_j^i.

In order to design a global controller for the TS fuzzy model (2) for the original nonlinear system (1), the parallel distributed compensation (PDC) technique [4] is adopted in this paper

Using the same premise as (2), the *i*th rule of the fuzzy logic controller (FLC) can be obtained as follows:

Controller Rule *i*:

$$\begin{aligned} &\text{IF } x_1(t) \text{ is } F_1^i \text{ and ... and } x_n(t) \text{ is } F_n^i \\ &\text{THEN } u = -K_i x \quad (i = 1, 2, \ldots, r) \end{aligned} \tag{4}$$

where K_i is the feedback gain vector. The fuzzy controller (4) is analytically represented by

$$\begin{aligned} u(t) &= \frac{\sum_{i=1}^{q} w_i(x(t))(-K_i x(t))}{\sum_{i=1}^{q} w_i(x(t))} = -\sum_{i=1}^{q} \mu_i(x(t))K_i x(t) \\ &= -K(\mu)x(t) \end{aligned} \tag{5}$$

The overall closed-loop fuzzy system obtained by combining (2) and (5) becomes

$$\dot{x} = \sum_{i=1}^{q} \sum_{j=1}^{q} \mu_i \mu_j (A_i - B_i K_j)x \tag{6}$$

A typical stability condition for fuzzy system (6) is given as follows:

Theorem 1. ([1]) The equilibrium of the fuzzy control system (6) is asymptotically stable in the large if there exists a common positive definite matrix **P** such that the following two inequalities are satisfied:

$$\{A_i - B_i K_i\}^T P + P\{A_i - B_i K_i\} < 0 \quad \text{for} \quad i = 1,2,\ldots,r \tag{7}$$

and

$$G_{ij}^T P + P G_{ij} < 0 \quad \text{for} \quad i < j \le r \tag{8}$$

where,

$$G_{ij} = \frac{A_i - B_i K_j + A_j - B_j K_i}{2}$$

Since the above stability condition is very conservative, the relaxed stability condition is proposed in [2].

Theorem 2. ([2]) The equilibrium of the fuzzy control system (6) is asymptotically stable in the large if there exists a common positive definite matrix **P** and **M** such that the following inequalities are satisfied:

$$P > 0, \quad M \ge 0 \tag{9}$$

$$G_{ii}^T P + P G_{ii} + (s-1)M < 0, \; (i = 1,2,\ldots,r) \tag{10}$$

$$\left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij} + G_{ji}}{2}\right) - M \le 0 \tag{11}$$

where s is the maximum of the number of fuzzy subsystems that are fired at an instant.

3. Proposed stability analysis method

In this section, an alternative approach for the analysis of the fuzzy-model-based controllers is detailed. Before going further we make the following assumption.

Assumption 1. The number of fuzzy rules, which are fired simultaneously for all, $t > 0$ is $s < r$.

In this case, we can define the subspace S_l ($l = 1,2,...s$) in the entire input space as the space where s rules are fired concurrently at an instant. The characteristic function of S_l is defined by

$$\eta_l = \begin{cases} 1 & x \in S_l \\ 0 & x \notin S_l \end{cases} \quad \sum_{l=1}^{s} \eta_l = 1 . \tag{12}$$

Define I_l as the set of indices of fuzzy rules in S_l. Then, on every subspace, the fuzzy system (2) can be denoted by

$$\begin{aligned} \dot{x} &= \overline{A}_l(t)x(t) + \overline{B}_l(t)u(t) \\ &= \sum_{i \in I_l} \mu(x(t))A_i + \sum_{i \in I_l} \mu(x(t))B_i u(t) , \quad x(t) \in S_l \end{aligned} \tag{13}$$

Therefore, the global system can be represented using (12) and (13) as follows:

$$\dot{x} = \sum_{l=1}^{m} \eta_l (\overline{A}_l(t)x(t) + \overline{B}_l(t)u(t)) \tag{14}$$

The system (14) can be viewed as a piecewise linear time-varying system. Actually system (14) is the piecewise linear combination of the smaller fuzzy systems. The above result is summarized in the following theorem.

Theorem 3. The fuzzy system (2) can be transformed to the piecewise linear time varying system (14), where each subsystem is the smaller fuzzy system if the number of fuzzy rules fired simultaneously for all, $t > 0$ is $s < r$.

There are some works on the stability analysis of the piecewise linear system [6-7], where piecewise linear quadratic (PQ) Lyapunov function is generally used. Therefore, in this work, the stability analysis of the fuzzy-model-based controller is studied with piecewise quadratic Lyapunov function as a tool for analyzing Lyapunov stability of the fuzzy-model-based controller. In order to use the PQ Lyapunov function, we first make the following assumption to guarantee the existence of the right-hand derivative and left-hand derivative.

Assumption 2. If l th subsystem is in the l th state space, it will stay in the l th state space for a period of time $\bar{t}$ where

$$\bar{t} > \tilde{\tau} , \quad \tilde{\tau} > 0 \text{ is a fixed constant}$$

and the number of traversing time instants among regions S_l is finite.

In this paper, we will use the following notion of stability for the fuzzy system (2).

Definition 1. The fuzzy system (2) (with $u = 0$) is said to be quadratically stable if there exists an $n \times n$ positive definite symmetric matrix P and a constant $\alpha > 0$ such that the Lyapunov derivative for the Lyapunov function $V(x) = x^T Px$ satisfies

$$L(x,t) = \frac{dV}{dt} = x^T P[A(\mu(x(t)))^T P + PA(\mu(x(t)))]x \le -\alpha \| x \|^2$$

for all pairs $(x,t) \in \Re^n \times \Re$. The fuzzy system (2) is said to be quadratically stabilizable if there exists a state feedback controller such that the closed-loop system is quadratically stable. It should be noted that a state feedback controller is not necessarily static controller.

First, we consider the autonomous fuzzy system ($u = 0$) as follows. We use PQ Lyapunov functions as a tool for analyzing Lyapunov stability of (14) with $u = 0$. Let

$$V_l = x^T P_l x \tag{15}$$

be a Lyapunov function for subset S_l. Then the global Lyapunov function can be constructed as

$$V = x^T Px = \sum_{l=1}^{m} \eta_l V_l = \sum_{l=1}^{m} \eta_l x^T P_l x \tag{16}$$

This kind of Lyapunov function widely used for the stability analysis of piecewise linear systems [3-7].

Lemma 1. The fuzzy system (2) (with $u = 0$) is quadratically stable if there exists symmetric matrices P_l such that

$$P_l > 0 \tag{17}$$

$$\overline{A}_l^T P_l + P_l \overline{A}_l^T < 0 , \quad (l = 1,2,...,m) \tag{18}$$

Proof: Let the Lyapunov function be (16). Then the time derivative of the Lyapunov function is

$$\dot{V} = \sum_{l=1}^{m}\eta_l\{x^T(\sum_{l=1}^{m}\eta_l\overline{A}_l^T)P + P\sum_{l=1}^{m}\eta_l(\sum_{l=1}^{m}\eta_l\overline{A}_l)x\}$$
$$= \sum_{l=1}^{m}\eta_l x^T(\overline{A}_l^T P + P\overline{A}_l)x \quad (19)$$

Therefore, if the inequalities (17) and (18) are satisfied, (19) is the fuzzy system (2) negative definite and the proof is completed.

If input exists we can obtain following theorem

Theorem 4. The fuzzy system (2) is quadratically stabilizable via fuzzy controller (4) if there exists symmetric matrices P_l such that

$$P_l > 0 \quad (20)$$
$$(\overline{A}_l - \overline{B}_l\overline{K}_l)^T P_l + P_l(\overline{A}_l - \overline{B}_l\overline{K}_l) < 0,\ (l = 1,2,\ldots,m) \quad (21)$$

where $\overline{K}_l = \sum_{i\in I_l}\mu_i(x(t))K_i$

Proof: Let the Lyapunov function candidate is (16). The closed loop fuzzy system is

$$\dot{x} = A(\mu)x(t) - B(\mu)K(\mu)x(t)$$
$$= \sum_{l=1}^{m}\eta_l\overline{A}_l x(t) - \sum_{l=1}^{m}\eta_l\overline{B}_l\sum_{l=1}^{m}\eta_l\overline{K}_l x(t)$$
$$= \sum_{l=1}^{m}\eta_l(\overline{A}_l - \overline{B}_l\overline{K}_l)x(t)$$

Using Lemma 1, the proof is completed.

Note that Cao et al. [3-5] used PQ Lyapunov function for the stability analysis of the fuzzy-model-based controller. The main difference of the Cao's method and the proposed method is as follows. In Cao's method, global fuzzy system is divided into uncertain linear systems where the interaction effect between fuzzy rules is regarded as uncertainty of the each linear system. On the other hand, we divide the global fuzzy system into fuzzy systems whose number of fuzzy rules is less than the former one. In this approach, the effect of the membership functions is implicitly considered.

4. Conclusion

In this paper, piecewise quadratic Lyapunov functions are used to analyze the stability of fuzzy-model-based controller. We represent the nonlinear system using a Takagi-Sugeno fuzzy model, which represent the given nonlinear system by fuzzy inference rules and local linear dynamic models. The proposed stability analysis technique is developed by dividing the whole fuzzy system into the smaller separate fuzzy systems to reduce the conservatism. Some sufficient conditions for the proposed method are obtained. Finally, stability of the closed system with various kinds of controller for TS fuzzy model is checked through the proposed method.

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