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DISCRETE-TIME MIXED H_2/H_∞ FILTER DESIGN USING THE LMI APPROACH

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Abstract: This paper deals with the optimal filtering problem constrained to input noise signal corrupting the measurement output for linear discrete-time systems. The transfer matrix H_2 and/or H_∞ norms are used as criteria in an estimation error sense. In this paper, the mixed H_2/H_∞ filtering problem in linear discrete-time systems is solved using the LMI approach, yielding a compromise between the H_2 and H_∞ filter designs. This filter design problems are formulated in a convex optimization framework using linear matrix inequalities. A numerical example is presented.

Keywords: Mixed H_2/H_∞ filtering, LMI(Linear matrix inequality), Convex optimization.

1 Introduction

The optimal filtering problem for linear discrete-time systems has been tackled in the past through the optimization of an H_2 norm criterion in an estimation error sense [1]. This kind of approach appears to be quite natural, since the knowledge of statistical properties of the input signal, particularly a white-noise process, corrupting the measurement output can be valued as the sum of the output variances which leads to the H_2 norm. The existence of an estimator structure, like the Kalman filter, associated with this problem is described through necessary and sufficient conditions associated with the solvability of a Riccati filtering equation [7], [11].

On the other hand, when sufficient details about the power spectral density of the input signal do not exist, the H_∞ performance criterion arises as an alternative strategy. In fact, the H_∞ norm of the transfer function from the input noise to the output is used to ensure a noise attenuation level for the estimation error dynamics [6], [10], [14].

The mixed H_2/H_∞ filtering problem for discrete-time systems has deserved less attention. The problem consists in minimizing an upper bound to the H_2 norm criterion while a prescribed noise attenuation level bounds the H_∞ norm [4], [5], stipulating a kind of trade-off between the H_2 performance and the noise attenuation [13]. If the prescribed attenuation level is set to nearly infinity, the optimal Kalman filter is obtained [12].

The discrete-time Riccati equation fulfils an important role in state-space approaches for the filtering problems. In fact, the filtering problem is solvable if and only if a Riccati equation admits a positive definite stabilizing solution [14]. The aim of this paper is to recast the H_2/H_∞ filtering problem in terms of linear matrix inequalities (LMIs). In this new framework, the global optimal solutions are attained through convex optimization procedures, which can be efficiently solved nowadays. Combining the two design techniques (H_2 and H_∞), the mixed H_2/H_∞ filter problem is solved. Using LMIs, a guaranteed H_∞ norm bound is imposed, while an upper bound to the H_2 norm is minimized. Similarly to the control problems [11], the optimal gain is obtained from the global solution of a convex optimizing procedure involving LMIs only.

The notation used is fairly standard : $E\{\cdot\}$ denotes the expectation. $\sigma_{max}(\cdot)$ denotes the maximum singular value of a matrix and ζ is the time-shift operator. The usual notation $\|\cdot\|_p$ stands for the H_p norm, in Hardy spaces.

2 Preliminaries and H_2/H_∞ Filtering Statements

Consider the following linear time invariant discrete-time system given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k, x(0) = x_0 \\ y_k &= Cx_k + Dw_k \\ z_k &= Lx_k \end{aligned} \tag{1}$$

where x_k is the state vector, y_k is the measurements output vector, w_K is the noise signal vector (including process and measurement noises) and z_k is the signal to be estimated. The initial state condition x_0 is considered to be known and without loss of generality it can be assumed to be zero [6]. Furthermore, the following standard assumption is made : $\{A, C\}$ is detectable. This assumption guarantees that there exists an observer constant gain such that the filter is asymptotically stable [3].

The key idea of the filtering problem is to find the estimate $\hat{z}_k$ of the signal z_k such that a performance criterion, such as H_2 or H_∞ norms, is minimized in an estimation error sense. The available estimates are based on the set of the measurement output signal obtained at each time k. In this sense, the purpose is to design an asymptotically stable linear filter described by

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + K(y - Cx_k),\ \hat{x}(0) = 0 \\ \hat{z}_k &= L\hat{x}_k \end{aligned} \tag{2}$$

where K is the filter constant gain to be determined. The filter has the standard Luenberger form. Assumption implies that K can be selected such that

$$A_c = A - KC \tag{3}$$

is asymptotically stable, that is all the A_c matrix eigenvalues are inside the open unit disc.

Defining the state error as $e_k = x_k - \hat{x}_k$, then the estimation error dynamics is given by

$$\begin{aligned} e_{k+1} &= A_c x_k + B_c w_k,\ e(0) = x(0) - \hat{x}(0) \\ \tilde{z}_k &= L e_k \end{aligned}$$

where $\tilde{z}_k = z_k - \hat{z}_k$ is the estimation error, and

$$B_c = B - KD \tag{4}$$

The closed-loop transfer function from the noise signal w_k to the output $\tilde{z}_k$ is given by

$$H_{zw}(\zeta) = L(\zeta I - A_c)^{-1} B_c \tag{5}$$

Its H_2 norm is defined by

$$\|H_{zw}\|_2^2 = \frac{1}{2}\int_{-\pi}^{\pi} Tr\{H_{zw}(e^{j\omega})^T H_{zw}(e^{j\omega})\}d\omega \tag{6}$$

and the H_∞ norm by

$$\|H_{zw}\|_\infty = \max_{\omega \in (\pi, -\pi)} \sigma_{max}\{H_{zw}(e^{j\omega})\}. \tag{7}$$

The H_2/H_∞ filtering design problem to be addressed in this paper is stated as follows : Determine a stable filter such that an upper bound to the H_2 performance criterion is minimized and $\|H_{zw}\|_\infty \leq \gamma$

3 Mixed H_2/H_∞ Filtering Problem

As stated above, the mixed H_2/H_∞ filter design deals with the problem of finding an estimation of z_k satisfying a prescribed noise attenuation level γ which also guarantees the minimization of an upper bound to the H_2 norm. In other words, the mixed H_2/H_∞ filter design claims to achieve a compromise between both performance criteria.

3.1 H_2 filtering problem

The H_2 norm of a strictly proper and stable transfer function $H_{zw}(\zeta)$, can be computed by

$$\|H_{zw}\|_2^2 = Tr\{B_c^T L_0 B_c\} \tag{8}$$

where L_0 is the observability Gramian obtained from the following discrete-time Lyapunov equation :

$$A_c^T L_0 A_c - L_0 + L^T L = 0 \tag{9}$$

Observe that A_c is asymptotically stable if and only if $L_0 \geq 0$ [15]. Matrices A_c and B_c are defined (3) and (4) respectively.

Define the following mapping Γ

$$\Gamma(Y) = A_c^T Y A_c - Y + L^T L \tag{10}$$

and the set

$$M_2 = \{Y | Y = Y^T \geq 0, \Gamma(Y) \leq 0\} \tag{11}$$

Clearly, any matrix $Y \in M_2$ is such that

$$Tr\{B_c^T Y B_c\} \geq Tr\{B_c^T L_0 B_c\} \tag{12}$$

since $Y \geq L_0 \in M_2$. The H_2 filtering problem is formulated and solved as follows [8] : *Consider the following optimization problem:*

$$\min_{J,Y,W} Tr\{J\} \tag{13}$$

subject to

$$\begin{bmatrix} J & B^T Y - D^T W^T \\ YB - WD & Y \end{bmatrix} \geq 0, \tag{14}$$

$$\begin{bmatrix} Y & YA - WC & 0 \\ A^T Y - C^T W^T Y & L^T & \\ 0 & L & 0 \end{bmatrix} \geq 0, \tag{15}$$

$$Y \geq 0 \tag{16}$$

The optimal solution is such that

$$Tr\{J\} = \min \|H_{zw}\|_2^2 \tag{17}$$

and the optimal H_2 filtering gain is given by

$$K = Y^{-1} W \tag{18}$$

3.2 H_∞ filtering problem

The construction of the H_∞ optimization problem for filtering can be formulated by the discrete-time bounded real lemma [2], [15] : A_c is asymptotically stable and $\|H_{zw}\|_\infty \leq \gamma$ if and only if there exists $X, X = X^T \geq 0$, satisfying

$$\begin{bmatrix} A_c & B_c \\ L & o \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ L & o \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} \leq 0.$$

The solution to H_∞ filtering problem given as follows [8] : *The optimal solution of*

$$\min_{Y,W,\gamma} \gamma^2 \tag{19}$$

subject to

$$\begin{bmatrix} Y & 0 & A^T Y - C^T W^T & L^T \\ 0 & \gamma^2 I & B^T Y - D^T W^T & 0 \\ YA - WC & YB - WD & Y & 0 \\ L & 0 & 0 & I \end{bmatrix} \geq 0$$

$$Y \geq 0 \tag{20}$$

where γ^2 is such that

$$\min \|H_{zw}\|_\infty = \gamma \tag{21}$$

and

$$K = Y^{-1} W \tag{22}$$

is the optimal H_∞ filtering gain.

3.3 Mixed H_2/H_∞ filtering problem

Combining the H_2 filtering problem with the H_∞ filtering problem, an LMIs characterization of the mixed H_2/H_∞ filtering is provided as follows :

The optimal solution of

$$\min_{J,Y,W} Tr\{J\} \tag{23}$$

subject to

$$\begin{bmatrix} J & B^TY - D^TW^T \\ YB - WD & Y \end{bmatrix} \geq 0 \tag{24}$$

$$\begin{bmatrix} Y & 0 & A^TY - C^TW^T & L^T \\ 0 & \gamma^2 I & B^TY - D^TW^T & 0 \\ YA - WC & YB - WD & Y & 0 \\ L & 0 & 0 & I \end{bmatrix} \geq 0$$

$$Y \geq 0 \tag{25}$$

with $Y = Y^T$ and $J = J^T$ is such that

$$Tr\{J\} \geq \|H_{zw}\|_2^2, \ \|H_{zw}\|_\infty \leq \gamma \tag{26}$$

and the optimal filtering gain is given by

$$K = Y^{-1}W \tag{27}$$

4 Example

Consider the following LTI discrete-time system described by

$$A = \begin{bmatrix} 0.242 & 0.544 & 0.216 \\ 0.726 & -0.188 & 0.883 \\ 0.199 & 0.231 & 0.233 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.308 & 0.313 \\ 0.933 & 0.362 \\ 0.215 & 0.292 \end{bmatrix},$$

$$C = \begin{bmatrix} 10 & 0 & 10 \end{bmatrix}, D = \begin{bmatrix} 0.566 & 0.483 \end{bmatrix},$$
$$L = \begin{bmatrix} 10 & 0 & 10 \end{bmatrix}.$$

The optimal H_2 filtering cost achieved is given by $Tr\{J\} = 91.841$ with

$$Y_2 = \begin{bmatrix} 131.527 & -55.231 & 135.038 \\ -55.231 & 100.916 & -64.175 \\ 135.038 & -64.175 & 140.816 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 8.291 \\ 2.109 \\ 8.028 \end{bmatrix},$$

and the optimal H_2 filtering constant gain is given by

$$K_2 = \begin{bmatrix} 0.057 \\ 0.075 \\ 0.037 \end{bmatrix}.$$

On the other hand, suppose that we have an arbitrary input noise with bounded energy. Then the H_∞ filtering design assures the optimal attenuation level, that is $\gamma^* = 9.601$ with the optimal observer gain

$$K_\infty = \begin{bmatrix} 0.054 \\ 0.068 \\ 0.034 \end{bmatrix}$$

obtained from the matrices

$$Y_\infty = \begin{bmatrix} 2854.703 & 328.752 & -4629.142 \\ 328.752 & 207.016 & -804.474 \\ -4629.142 & -804.474 & 8562.316 \end{bmatrix},$$

$$W_\infty = \begin{bmatrix} 17.805 \\ 4.318 \\ -11.302 \end{bmatrix}.$$

In fact, this is the optimal attenuation level since, computing the H_∞ norm of the estimation error system with the constant gain K_∞, $\|H_{zw}\|_\infty = 9.607$ is obtained. The computed H_2 norm of the estimation error system for K_∞ is given by $\|H_{zw}\|_2^2 = 92.287$.

Taking now the prescribed noise attenuation level $\gamma = 11.53$, let us find an upper bound to the H_2 norm such that $\|H_{zw}\|_\infty \leq \gamma$. Using the result (26), the optimal cost $Tr\{J\} = 103.5903$ is obtained, which is associated with the mixed H_2/H_∞ constant gain

$$K_{2\infty} = \begin{bmatrix} 0.053 \\ 0.071 \\ 0.034 \end{bmatrix}$$

from the matrices

$$Y_{2\infty} = \begin{bmatrix} 132.599 & -61.539 & 134.779 \\ -61.539 & 121.518 & -71.049 \\ 134.779 & -71.049 & 142.546 \end{bmatrix},$$

$$W_{2\infty} = \begin{bmatrix} 7.264 \\ 2.924 \\ 6.971 \end{bmatrix}.$$

Note that the H_2 norm values associated with the filter constant gain K is $\|H_{zw}\|_2^2 = 92.2581 \leq Tr\{J\}$ and, moreover, $\|H_{zw}\|_\infty = 9.8132 \leq \gamma$.

Considering $x(0) = [0.5 \ -0.3 \ 0.2]^T$ as the initial state vector, the estimation errors are depicted in Figure 1 (a), (b) and (c) respectively for the H_2, H_∞ and H_2/H_∞ filtering design.

5 Conclusions

Optimal filtering problems for linear time invariant discrete-time systems with H_2 and H_∞ norm criteria are parameterized in terms of convex optimization procedures have been described by LMIs. It is shown that the mixed H_2/H_∞ filtering problem can be formulated and solved by combining the solution of the H_2 filter with that of the H_∞ filter. Since powerful numerical tools are suitable to deal with problems in an LMI setting, the practical computing character of the techniques developed here arises immediately.

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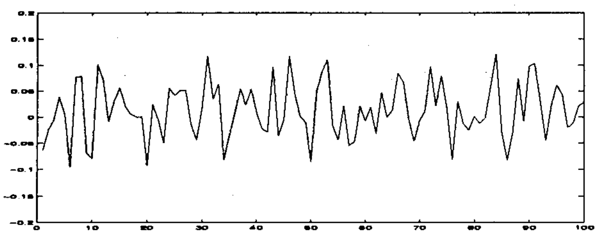
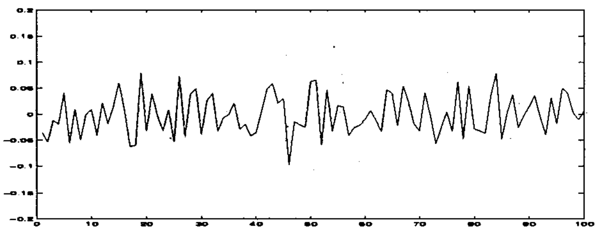
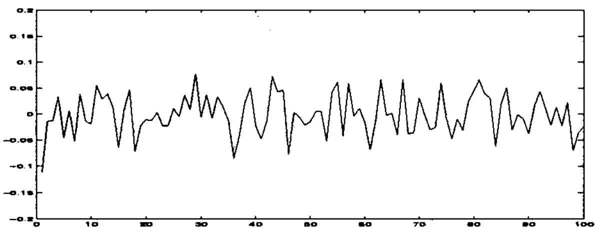


(a) H_2 Filtering



(b) H_∞ Filtering



(c) H_2/H_∞ Filtering

Figure 1: Estimation Errors

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