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Stability of discrete state delay systems

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Abstract

A new method to solve a Lyapunov equation for a discrete delay system is proposed. Using this method, a Lyapunov equation can be solved from a simple linear equation and N-th power of a constant matrix, where N is the state delay. Combining a Lyapunov equation and frequency domain stability, a new stability condition is proposed. The proposed stability condition ensures stability of a discrete state delay system whose state delay is not exactly known but only known to lie in a certain interval.

1 Introduction

State delays are frequently encountered in control problems of many physical systems. In particular, continuous state delay systems have been received a lot of attentions and many stability results have been proposed (see [1] and its references). On the other hand, there has been less attention to the following discrete state delay system

$$x(k+1) = A_0 x(k) + A_1 x(k-N) \tag{1}$$

where $x \in \mathbb{R}^n$ is a state. The reason of less attention is not surprising since system (1) can be transformed into an equivalent non-delayed system. Introducing an augmented new state $z(k)$ as follows:

$$z(k) \triangleq \left[\begin{array}{c} x(k) \\ \hline x(k-1) \\ \vdots \\ x(k-N) \end{array}\right] \in \left[\begin{array}{c} \mathbb{R}^{n\times 1} \\ \hline \mathbb{R}^{nN\times 1} \end{array}\right],$$

we can obtain an equivalent non-delayed system

$$z(k+1) = \mathcal{A}_N z(k) \tag{2}$$

where

$$\mathcal{A}_N \triangleq \left[\begin{array}{c|ccc} A_0 & & & A_1 \\ \hline I & & & \\ & \ddots & & \\ & & I & \end{array}\right].$$

Now stability of (1) can be investigated by simply checking stability of an ordinary non-delayed system (2).

However, there are two important cases that stability check of the equivalent non-delayed system is not adequate for stability check of (1). The first case is when the state delay N is very large, and the second case is the state delay N is not exactly known but only known to lie in a certain interval. If N is very large, then the $\mathcal{A}_N$ matrix of (2) is very large (note $\mathcal{A}_N \in \mathbb{R}^{n(N+1)\times n(N+1)}$). Hence stability check of $\mathcal{A}_N$ is numerically demanding and sometimes an unstable task. If the state delay N is only known to lie in a certain interval (for example, $N \in [0, N_{\max}]$), then stability should be checked for each $N \in [0, N_{\max}]$, which is also a numerically demanding task, in particular for large $N_{\max}$.

To cope with these two cases, we propose a new non-conservative stability condition of (1) by carefully investigating a Lyapunov equation for (2). In Section 2, it is shown that a solution of a Lyapunov equation for (2) can be transformed into a simple linear equation, where the only term depending on N is N-th power of a constant matrix. Hence, even for large N, the computation is simple. In Section (3), combining the constant matrix with frequency domain interpretations, we propose a stability condition for (1), where the state delay N is only known to lie in a certain interval. In Section (4), a numerical example is given to illustrate the results of this paper. In Section (5), conclusion is provided.

The works which are most related to ours are [2, 3, 4, 5]. In [2, 3, 4], so-called delay independent stability conditions are considered. The conditions are, however, conservative when the state delay is known to lie in a certain interval. In [5], a robust stability problem is considered for a exactly known delay case.

Notation is standard. For a matrix $M \in \mathbb{C}^{n\times n}$ given by

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix},$$

$\operatorname{cs} M$ is defined by

$$\operatorname{cs} M \triangleq [\, m_{11} \ \cdots \ m_{n1} \mid \cdots \mid m_{1n} \ \cdots \ m_{nn} \,]' \in \mathbb{C}^{n^2\times 1}$$

Symbols $\mathcal{N}$, $\mathcal{R}$ and $\otimes$ denote null space, range space and kronecker product, respectively.

2 Lyapunov function

The Lyapunov function for (2) is defined by

$$V(z(k)) \triangleq z(k)'\mathcal{P}z(k),$$

where symmetric matrix $\mathcal{P} \in \mathbb{R}^{n(N+1)\times n(N+1)}$ is partitioned compatible to the partition of $z(k)$ and labeled as follows:

$$\mathcal{P} = \left[\begin{array}{c|ccc} P_{00} & P_{01}(N) & \cdots & P_{01}(1) \\ \hline P_{10}(N) & P_{11}(N,N) & \cdots & P_{11}(N,1) \\ \vdots & \vdots & \ddots & \vdots \\ P_{10}(1) & P_{11}(1,N) & \cdots & P_{11}(1,1) \end{array}\right].$$

It is standard from the following

$$V(z(k+1)) - V(z(k)) = z(k)'(\mathcal{A}_N'\mathcal{P}\mathcal{A}_N - \mathcal{P})z(k),$$

that the system (2) is stable if and only if there exists $\mathcal{P} = \mathcal{P}' > 0$ satisfying $\mathcal{A}_N'\mathcal{P}\mathcal{A}_N - \mathcal{P} < 0$. From the special structure of $z(k)$ ($z(k)$ is stable if and only if $x(k)$ is stable), the condition $\mathcal{A}_N'\mathcal{P}\mathcal{A}_N - \mathcal{P} < 0$ can be modified as in the following lemma.

Lemma 1 *System (1) is stable if and only if there exists $\mathcal{P} = \mathcal{P}' > 0$ satisfying*

$$\mathcal{A}_N'\mathcal{P}\mathcal{A}_N - \mathcal{P} + \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right] = 0 \tag{3}$$

for some $Q = Q' \in \mathbb{R}^{n\times n} > 0$.

Matrix $\mathcal{P}$ has 3 variables P_{00}, $P_{01}(i) = P_{10}(i)'$ and $P_{11}(i,j) = P_{11}(j,i)'$. The following lemma simplifies the expression of $\mathcal{P}$ using one variable $X(i)$.

Lemma 2 *The solution $\mathcal{P}$ satisfying (3) is given by*

$$\begin{aligned} P_{00} &= X(0), \\ P_{01}(i) &= X(i)A_1, \\ P_{11}(i,j) &= \begin{cases} A_1'X(i-j)'A_1, & 0 \le j \le i \le N \\ A_1'X(j-i)A_1, & 0 \le i \le j \le N, \end{cases} \end{aligned} \tag{4}$$

where $X(k)$, $0 \le k \le N$ is given by

$$\begin{gathered} A_0'X(0)A_0 + A_1'X(N)'A_0 + A_0'X(N)A_1 + \\ A_1'X(0)A_1 - X(0) + Q = 0, \\ X(0) = X(0)', \\ X(k+1) = A_0'X(k) + A_1'X(N-k)',\ 0 \le k \le N-1. \end{gathered} \tag{5}$$

Since the matrix difference equation (third equation) in (5) is not in an easy form to solve, the matrix difference equation is transformed into a kind of two point boundary value problem in the next lemma. Throughout the paper, A_0 is assumed to be nonsingular: we note that most discrete systems have nonsingular A_0 matrices.

Lemma 3 *The matrix difference equation in (5) is equivalent to the following.*

$$\begin{bmatrix} \operatorname{cs} X(k+1) \\ \operatorname{cs} X(N-k-1) \end{bmatrix} = \begin{bmatrix} A & B \\ -A^{-1}BA & A^{-1}(I-BB) \end{bmatrix} \begin{bmatrix} \operatorname{cs} X(k) \\ \operatorname{cs} X(N-k) \end{bmatrix} \tag{6}$$

where $A \triangleq (I \otimes A_0')$, $B \triangleq (I \otimes A_1')T$, and

$$T \triangleq [\, T_1 \mid T_2 \mid \cdots \mid T_{n^2} \,],\ T_l \in \mathbb{R}^{n^2\times 1}. \tag{7}$$

Row vector T_l, $1 \le l \le n^2$ is defined by

$$T_{(i-1)n+j} \triangleq e_{(j-1)n+i},\ 1 \le i,j \le n,$$

where $e_l \in \mathbb{R}^{n^2\times 1}$, $1 \le l \le n^2$ is a row vector whose l-th element is 1 and all other elements are 0.

For later reference, we define two matrices H (see (6)) and J:

$$H \triangleq \begin{bmatrix} A & B \\ -A^{-1}BA & A^{-1}(I-BB) \end{bmatrix},\ J \triangleq \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{8}$$

Now return to the problem of solving (5) for some Q. The matrix Lyapunov equation (first equation) and the matrix difference equation (third equation) are coupled in (5). To solve the Lyapunov equation, it is necessary to obtain an $X(0)$ and $X(N)$ pair satisfying the matrix difference equation, or equivalently (6). The constraint imposed on any $X(0)$ and $X(N)$ pair satisfying (6) can be stated using the boundary condition:

$$\begin{bmatrix} \operatorname{cs} X(N) \\ \operatorname{cs} X(0) \end{bmatrix} = H^N \begin{bmatrix} \operatorname{cs} X(0) \\ \operatorname{cs} X(N) \end{bmatrix},$$

and the above equation can be expressed using J (see (8)) as follows:

$$(I - JH^N) \begin{bmatrix} \operatorname{cs} X(0) \\ \operatorname{cs} X(N) \end{bmatrix} = 0. \tag{9}$$

Does an $X(0)$ and $X(N)$ pair satisfying (9) always exist? For example, if $\dim \mathcal{N}(I - JH^N) = 0$, then only $[(\operatorname{cs} X(0))'\ (\operatorname{cs} X(N))']' = 0$ can be the solution to (9). The next lemma shows that $\dim \mathcal{N}(I - JH^N) = n^2$ and thus there exists a nontrivial $X(0)$ and $X(N)$ pair satisfying (9).

Lemma 4 *The following is satisfied.*

$$\dim \mathcal{N}(I - JH^N) = n^2. \tag{10}$$

The proof of Lemma 4 needs the following lemma.

Lemma 5 *If z is an eigenvalue of H, then z^{-1} is also an eigenvalue of H.*

From the proof of Lemma 5, we note that eigenvalues and eigenvectors of H (for simplicity, all eigenvalues of H are assumed to be simple and nonzero) can be expressed as

$$H\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \Sigma^{-1} \end{bmatrix}, \quad (11)$$

where $\Sigma \in \mathbb{C}^{n^2\times n^2}$ is a diagonal matrix whose diagonal elements are eigenvalues of H.

PROOF of Lemma 4. First we will show that

$$\dim \mathcal{N}(I - JH^N) \geq n^2. \quad (12)$$

Let v is defined by

$$v \triangleq \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\begin{bmatrix} I \\ \Sigma^N \end{bmatrix}\alpha,$$

where $\alpha \in \mathbb{C}^{n^2\times 1}$, then $v \in \mathcal{N}(I - JH^N)$ since from (11)

$$(I - JH^N)\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\begin{bmatrix} I \\ \Sigma \end{bmatrix}\alpha = \begin{bmatrix} X + Y\Sigma^N \\ Y + X\Sigma^N \end{bmatrix}\alpha - \begin{bmatrix} X + Y\Sigma^N \\ Y + X\Sigma^N \end{bmatrix}\alpha = 0$$

for all $\alpha \in \mathbb{C}^{n^2\times 1}$. Hence we have

$$\dim \mathcal{N}(I - JH^N) \geq \dim \mathcal{R}(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\begin{bmatrix} I \\ \Sigma^N \end{bmatrix}) = n^2.$$

Similarly, we can show that

$$\dim \mathcal{N}(-I - JH^N) \geq n^2. \quad (13)$$

From (12) and (13), we can conclude that JH^N has only two eigenvalues ± 1 and thus $\dim \mathcal{N}(I - JH^N) = 2n^2 - \mathcal{N}(-I - JH^N) \leq n^2$. From (12), we obtain (10).

From (10), the singular value decomposition of $(I - JH^N)$ is given by

$$(I - JH^N) = U\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}V^*,$$

where U, V are unitary matrices, and $\Sigma_1 \in \mathbb{R}^{n^2\times n^2}$ is a diagonal matrix whose diagonal elements are nonzero singular values of $(I - JH^N)$. Thus the constraint on an $X(0)$ and $X(N)$ pair satisfying (6) is given by

$$[\, R_1 \; R_2 \,] \triangleq [\, \Sigma_1 \; 0 \,]V^*\begin{bmatrix} \mathrm{cs}\, X(0) \\ \mathrm{cs}\, X(N) \end{bmatrix} = 0. \quad (14)$$

Using (14), the coupled equations (5) can be reduced to a simple linear equation.

Theorem 1 *An $X(0)$ and $X(N)$ pair satisfying (5) can be computed by the following equation:*

$$\begin{bmatrix} (1,1) & (1,2) \\ R_1 & R_2 \end{bmatrix}\begin{bmatrix} \mathrm{cs}\, X(0) \\ \mathrm{cs}\, X(N) \end{bmatrix} = \begin{bmatrix} -\mathrm{cs}\, Q \\ 0 \end{bmatrix}, \quad (15)$$

where R_1 and R_2 are from (14), and

$$\begin{aligned} (1,1) &\triangleq (A_0' \otimes A_0') + (A_1' \otimes A_1') - I \\ (1,2) &\triangleq (A_0' \otimes A_1')T + (A_1' \otimes A_0'). \end{aligned}$$

Remark 1 *Once $X(0)$ and $X(N)$ are obtained, $X(i)$, $2 \leq i \leq N-1$ can be computed easily from (6). For example, $X(1)$ and $X(N-1)$ are computed by*

$$\begin{bmatrix} \mathrm{cs}\, X(1) \\ \mathrm{cs}\, X(N-1) \end{bmatrix} = H\begin{bmatrix} \mathrm{cs}\, X(0) \\ \mathrm{cs}\, X(N) \end{bmatrix}.$$

Hence the Lyapunov equation (3) can be solved from a simple linear equation (15) and up to N-th power of the constant matrix H.

3 Stability condition

In this section, we show that eigenvalues and eigenvectors of H are closely connected with frequency domain stability of (1). Based on this observation, we propose a new stability condition which ensure stability of (1) for all $N \in [0, N_{\max}]$.

System (1) is stable if and only if $x(k+1) = A_0'x(k) - A_1'x(k-N)$ is stable. Thus (1) is stable if and only if the characteristic equation

$$\det(zI - A_0' - A_1'z^{-N}) = 0 \quad (16)$$

has all its roots inside the unit circle. Now suppose that (1) is stable for $N = 0$ (i.e., all roots of (16) lie inside the unit circle) and unstable $N = N_{\max}$ (i.e., at least one root of (16) is not inside the unit circle). Then since a root of

$$\det(zI - A_0' - A_1'z^{-r}) = 0, \; r \in \mathbb{R} \geq 0 \quad (17)$$

varies continuously with respect to the change of r, there exists $\bar{r} \in (0, N_{\max}]$ such that (17) has a root on the unit circle. From this observation, we obtain the following lemma.

Lemma 6 *If (1) is stable for $N = 0$ and (17) does not have a root on the unit circle for all real number $r \in [0, N_{\max}]$, then (1) is stable for $N \in [0, N_{\max}]$.*

The next theorem shows that unit circle roots of (17) can be checked from eigenvalues of H.

Theorem 2 *If (17) has a unit circle root, then the root is an eigenvalue of H.*

Using Theorem 2, we can compute $N_{\max}$ such that (1) is stable for all $N \in [0, N_{\max}]$.

Lemma 7 *Let e^{jw_i}, $w_i \in \mathbb{R} \geq 0$ be a unit circle eigenvalue of H and v_i be the corresponding eigenvector. Let $r_i \in \mathbb{R} \geq 0$ be defined by*

$$r_i \triangleq \begin{cases} \left|\mathrm{Im}(\ln(\frac{k\text{-th element of } v_i}{(n^2+k)\text{-th element of } v_i}))\right| / w_i, & w_i \neq 0 \\ 0, & w_i = 0, \end{cases} \quad (18)$$

where $k \leq n^2$ can be chosen arbitrarily as long as k-th element of v_i is nonzero. If (1) is stable for $N = 0$ and $N_{\max}$ is the greatest integer not larger than $\min r_i$, then (1) is stable for all $N \in [0, N_{\max}]$.

One extreme case is $N_{\max} = \infty$, that is (1) is stable for all $N \geq 0$. In this case, system (1) is called to be *delay independently stable* [2, 3].

Lemma 8 *If (1) is stable for $N = 0$ and H does not have a unit circle eigenvalue, then (1) is stable for all $N \geq 0$.*

4 Numerical example

Consider the following system

$$x(k+1) = \begin{bmatrix} 0.3 & 0.15 \\ 0 & 0.7 \end{bmatrix} x(k) + \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & -0.4 \end{bmatrix} x(k-N). \quad (19)$$

The system is stable for $N = 0$. A and B matrices of H are given by

$$A = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0.15 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0.15 & 0.7 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 & 0 & 0.1 & 0 \\ -0.2 & 0 & -0.4 & 0 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & -0.2 & 0 & -0.4 \end{bmatrix},$$

respectively. Eigenvalues of H are given by

$$\{0.2919, 0.3012, 0.6826, 0.9721 \pm j0.2346, 1.4650, 3.3205, 3.4256\}.$$

Note that there exists a unit circle root $0.9721 + j0.2346 = e^{jw}$, $w \geq 0$, and $w = 0.2368$. The corresponding eigenvector v is given by

$$v = [\ -0.0784,\ -0.2921,\ -0.1687,\ -0.6155,\ -0.0189,\ 0.1988,\ -0.0299,\ -0.678\ ]'.$$

From (18), we obtain $r = 10.2483$, and thus $N_{\max} = 10$. Hence we can conclude that (19) is stable for $N \in [0, 10]$. In fact, by checking $\mathcal{A}_N$, we can verify that (19) is stable for $N \leq 10$ and unstable $N = 11$.

5 Conclusion

In this paper, we have proposed an easy method to solve a Lyapunov equation for a state delay system. Using this method, we can solve a Lyapunov equation even for large N without causing numerical problems. Based on the relationship between frequency domain stability and a constant matrix that appears in a Lyapunov equation, we have proposed a new stability condition. The proposed stability condition ensures stability of a discrete state delay system whose state delay is not exactly known but known to lie in a certain interval.

References

[1] S.-I. Niculescu, E. I. Verriest, L. Dugard, and J.-M. Dion, "Stability of linear systems with delayed state: A guided tour," in *Proceedings of IFAC International Workshop on Linear Time Delay Systems*, (France), pp. 37–44, 1998.

[2] J. W. Wu and K.-S. Hong, "Delay-independent exponential stability criteria for time-varying discrete delay systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 4, pp. 811–814, 1994.

[3] H. Trinh and M. Aldeen, "D-stability analysis of discrete-delay perturbed systems," *Int. J. Contr.*, vol. 61, no. 2, pp. 493–505, 1995.

[4] S.-H. Song and J.-K. Kim, "H_∞ control of discrete-time linear systems with norm-bounded uncertainties and time delay in state," *Automatica*, vol. 34, no. 1, pp. 137–139, 1998.

[5] H. Trinh and M. Aldeen, "Robust stability of singularly perturbed discrete-delay systems," *IEEE Trans. Automat. Contr.*, vol. 40, no. 9, pp. 1620–1623, 1995.

[6] M. Hahn, *Theory and Application of Lyapunov's Direct Method.* Englewood Cliffs, N.J: Prentice Hall, 1963.