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Nonlinear Modification Scheme for Reducing Cautiousness of Linear Robust Control

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Abstract

In this paper, we develope a composite control law for linear systems with norm-bounded time-varying parameter uncertainties, which consists of a basic linear robust control designed so as to generate a desired transient time-response for the worst-case parameter variation and a nonlinear modification term designed so as to reduce cautiousness of the linear robust control in an adaptive manner. The proposed controller is established such that the reduction of cautiousness of the linear robust control is well incorporated into the achievement of a good transient behavior.

Key words - linear robust control, norm-bounded uncertainty, cautiousness, transient time-response, nonlinear modification, adaptation mechanism.

1. Introduction

For linear systems with norm-bounded parameter uncertainties, a large number of quadratic stabilizing linear control laws have been developed(e.g. see [8],[11]), and the connection between quadratic stabilizing control and H_∞ control has also been investigated(e.g. see [3]). In particular, recent literatures address the problem of designing quadratic stabilizing control considering additional performance robustness in terms of quadratic cost [4],[9],[10], H_∞ disturbance attenuation [12], closed-loop pole location [1] and so on. In these methods, the same level of performance robustness is equally required for systems corresponding to all possible parameter variations, and therefore they result in the worst-case design. In the actual situations, however, there are hardly the cases where the worst-case parameter variation occurs at all times. In this sense, linear robust control is cautious and intrinsically defensive strategy.

To deal with this problem, we focus on the potentiality of adaptation. Of course, for the plant considered here, conventional schemes for parameter estimation have no effective way. Very recently, for this kind of problems, several efforts [5],[6],[7],[13] including the author's works have been made to establish quadratic stabilizing control with certain adaptation mechanism for achieving higher transient performance. The underlying concept common to these methods is the active utilization of a posteriori information about system uncertainties under quadratic stability constraint.

In this paper, we propose an "active robust" controller design method for improving transient performance, which has a composite form consisting of a couple of "passive robust" controllers. First, we consider a typical pattern of worst-case parameter variation assumed implicitly in designing a linear robust control. Second, a basic linear robust control is designed such that the closed-loop system with the worst-case parameter exhibits a satisfactory transient performance. Finally, additional nonlinear control is designed such that the error between the state trajectory of the plant and that of a model with the worst-case parameter variation is to be minimized, which means the reduction of cautiousness of the linear robust control, and also results in the improvement of transient behavior. The effectiveness of this method is shown through some numerical examples.

2. Problem statement

Consider the following linear multi-input multi-output system described by the state equation:

$$\dot{x}(t) = (A + D\Delta(t)E)x(t) + Bu(t),\ x(0)=x_0 \qquad (1)$$
$$\Delta(t) \in \mathcal{B}_\Delta$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$ are vectors of the state to be regulated and the control input, respectively. The matrix A is the nominal system matrix. The time-varying matrix $\Delta(t)$ represents the norm-bounded parameter uncertainties belonging to the set

$$\mathcal{B}_\Delta := \{\Delta : [0, +\infty) \mapsto \mathrm{R}^{p\times q} \mid \|\Delta(t)\| \leq 1,\ \forall t \geq 0\} \qquad (2)$$

where the notation $\|\cdot\|$ denotes a matrix norm induced by the standard Euclidean norm of a vector i.e., $\|\Delta\| = \sqrt{\lambda_{\max}(\Delta^T\Delta)}$. The constant matrices D and E represent the structure of uncertainties.

We first recall the well-known results about linear robust control.

Definition (Quadratic Stability): The system (1) with $u = 0$ is said to be quadratically stable if there exists a fixed positive definite matrix $P > 0$ such that $V(x) = x^T Px$ is a Lyapunov function, i.e. for a scalar $\alpha > 0$,

$$\dot{V}(x) = 2x^T P(A + D\Delta(t)E)x \leq -\alpha\|x\|^2$$

along the trajectories of (1) corresponding to all possible parameter variations $\Delta(t) \in \mathcal{B}_\Delta$. The system is said to be quadratically stabilizable by linear control if there exists a state-feedback control $u = Kx$ such that the closed-loop system is quadratically stable.

Theorem 1 : The following statements are equivalent:
(a) The system (1) is quadratically stabilizable by linear state-feedback control; there exist $P > 0$ and K such that

$$(A + D\Delta(t)E + BK)^T P + P(A + D\Delta(t)E + BK) < 0 \quad \text{for all } \Delta(t) \in \mathcal{B}_\Delta \tag{3}$$

(b) There exist $P > 0$,K and $\varepsilon > 0$ satisfying the algebraic matrix equality

$$(A + BK)^T P + P(A + BK) + \varepsilon PDD^T P + \frac{1}{\varepsilon} E^T E + Q + K^T RK = 0 \tag{4}$$

for any $Q \geq 0$ and $R > 0$.
(proof) (a)$\Leftarrow$ (b) is obvious. (a)$\Rightarrow$ (b): The condition (a) implies the existence of $P' > 0$ and K satisfying

$$(A + D\Delta(t)E + BK)^T P' + P'(A + D\Delta(t)E + BK) + Q + K^T RK < 0 \quad \text{for all } \Delta(t) \in \mathcal{B}_\Delta$$

, and therefore implies (b) (e.g. see [8],[9],[10]). ■
Remark 1: If the above matrix equality (4) has a solution (P, K), and $u = Kx$ is applied to the system (1), it readily follows that the quadratic cost function

$$J(x_0, \Delta, u) = \int_0^\infty (x^T Qx + u^T Ru)dt \tag{5}$$

evaluated along the trajectories of the system (1) has the upper bound of the form:

$$J(x_0, \Delta, Kx) \leq x_0^T Px_0 \quad \text{for all } \Delta(t) \in \mathcal{B}_\Delta$$

The control input minimizing $\mathrm{tr}(P)$ is given as

$$u(t) = Kx(t) = -R^{-1}B^T Px(t) \tag{6}$$

where $\mathrm{tr}(P)$ represents the average cost bound when the initial value x_0 is assumed to be zero-mean random variable with the covariance $E(x_0 x_0^T) = I$. By substituting (6) into (4), we have the following algebraic Ricatti equation:

$$A^T P + PA + P\left(\varepsilon DD^T - BR^{-1}B^T\right)P + Q + \frac{1}{\varepsilon}E^T E = 0 \tag{7}$$

It is shown in [8] that whether the Ricatti equation (7) has a solution or not is independent of the choice of weighting matrices Q and R. ■

In (4) or (7), the term $\varepsilon PDD^T P + \frac{1}{\varepsilon}E^T E$ plays a key role in ensuring stability robustness, which essentially comes from the following inequality for the unknown part depending on $\Delta(t)$ appeared in computing $\frac{\mathrm{d}}{\mathrm{dt}}(x^T Px)$.

$$\begin{aligned} & x^T PD\Delta(t)Ex + x^T E^T \Delta^T(t) D^T Px \\ \leq\ & 2\, \|D^T Px\| \cdot \|\Delta(t)Ex\| \\ \leq\ & 2\, \|D^T Px\| \cdot \|Ex\| && (8) \\ \leq\ & x^T \left(\varepsilon PDD^T P + \frac{1}{\varepsilon}E^T E\right) x && (9) \end{aligned}$$

However, depending on both the structure and the actual pattern of parameter uncertainty $\Delta(t)$, the bound (8) or (9) may become an excessive overbound in terms of both sign and magnitude, and the corresponding robust control (6) may result in an overly cautious one. In this paper, we present a modification scheme for reducing the degree of cautiosness of linear robust control, which lead to the achievement of a desired transient time-response. Our main focus is on additional adaptive action to linear robust control. It should be noted that robust control framework is intrinsically passive, and can not address the problem of improving transient performance in an active manner. In the next section, we give a basic concept of cautiousness of linear robust control.

3. Cautiousness of linear robust control

Now, it is supposed that a linear robust control is obtained as the form of (6), in which the matrix P is the positive definite solution to (7) for appropriately selected $Q \geq 0$ and $R > 0$. Here, we define the following set $\mathcal{W}_\Delta$ consisting of the worst-case parameter variations assumed implicitly in designing the robust control (6).

$$\mathcal{W}_\Delta := \{\Delta : [0, +\infty) \mapsto \mathrm{R}^{p\times q} \mid x^T PD\Delta(t)Ex = \|D^T Px\|\cdot\|Ex\|,\ \forall t \geq 0\} \tag{10}$$

where $x(t)$ is the state trajectory of the system (1) with the control (6). If $D^T Px \neq 0$ and $Ex \neq 0$, then $\Delta(t) \in \mathcal{W}_\Delta$ implies $\|\Delta(t)\| = 1$. As a typical pattern of worst-case parameter variation belonging to the set $\mathcal{W}_\Delta$, we consider the so-called "rank-1" type, i.e.,

$$\widehat{\Delta}(t) = \begin{cases} \frac{1}{\|D^T Px\|\|Ex\|} D^T Pxx^T E^T & \text{if } D^T Px \neq 0 \text{ and } Ex \neq 0 \\ \widehat{\Delta}(t^-) & \text{otherwise} \end{cases} \tag{11}$$

where $t^- = \lim_{\tau>0,\tau\to 0}(t - \tau)$, for which $\mathrm{rank}(\widehat{\Delta}(t)) = 1$ and $\|\widehat{\Delta}(t)\| = 1$ hold at any time. In the case of $D^T Px_0 = 0$ or $Ex_0 = 0$, $\widehat{\Delta}(0)$ is given appropriately such that $\|\widehat{\Delta}(0)\| = 1$. Setting $\widehat{\Delta}(t) = \widehat{\Delta}(t^-)$ is to maintain piecewise continuity of $\widehat{\Delta}(t)$. Furthermore, using (11) we introduce the following model:

$$\dot{\widehat{x}}(t) = (A + D\widehat{\Delta}(t)E + BK)\widehat{x}(t), \quad \widehat{x}(0) = x_0 \tag{12}$$

On the other hand, the actual closed-loop system using the linear robust control (6) can be written as

$$\dot{x}(t) = (A + D\Delta(t)E + BK)x(t) \tag{13}$$

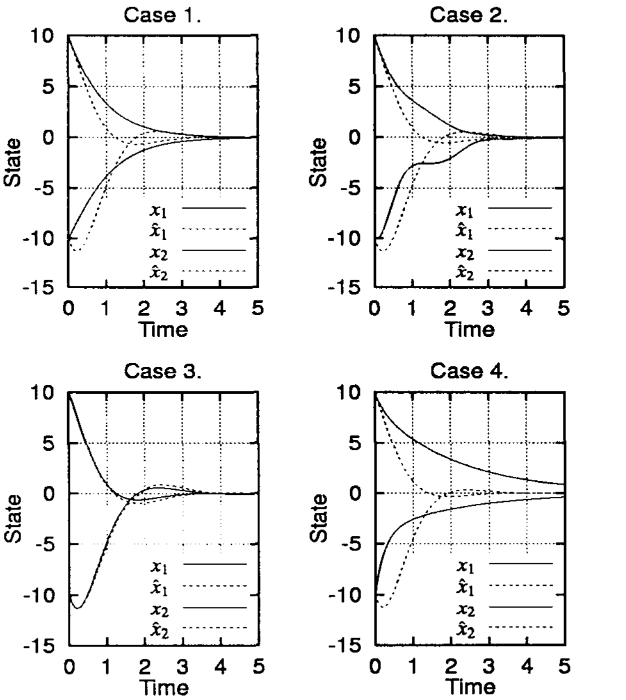
While both these trajectories are ensured to be quadratically stable, the appreciable error $e = x - \widehat{x}$ might be observed when $\|\Delta - \widehat{\Delta}\|$ is not negligible. In this case, the linear robust control should be considered to be overly cautious. From the above view point, the degree of cautiousness of the linear robust control could be reduced by modifying it so as to minimize the error. Of course, the degree of cautiousness of the linear robust control (6) depends on which pattern is selected out of the set $\mathcal{W}_\Delta$ as a representative of the worst-case variation. Our aim in this paper is to achieve a desired transient performance using a "moderate" control without excessive cautiousness.

Example 1: Here, we show a numerical example in which the differences in the transient time-response indicate the cautiousness of the linear robust control. The feedback gain K given below is determined so as to generate a desired state trajectory for the nominal system; The solid curve in the Case 1 exhibits the desired time-response.

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\ E = I$$

$$\varepsilon = 0.25,\ Q = I,\ R = 1,\ P = \begin{pmatrix} 9.673 & 1.099 \\ 1.099 & 3.794 \end{pmatrix}$$

$$K = (-1.099, -3.794),\ x_0 = (10, -10)$$



Case 1: $\Delta(t) \equiv 0$, Case 2: $\Delta(t) = \mathrm{diag}(\sin\pi t, \cos\pi t)$
Case 3: $\Delta(t) \equiv \mathrm{diag}(-1, 1)$, Case 4: $\Delta(t) = -\widehat{\Delta}(t)$;

4. Basic linear robust control and nonlinear modification

In this section, we develope a composite control law of the form

$$u(t) = K_B x(t) + v(t) \tag{14}$$

which consists of a basic linear robust control $K_B x$ and an additional nonlinear modification term v for reducing cautiousness of the basic linear control.

First, a basic linear robust control is determined over the class of quadratic stabilizing control such that a desired transient time-response is obtained under the assumption that the worst-case parameter given by (11) occures at all times. The key idea is that desired transient response to follow is specified as the trajectory of the system with worst-case parameter variation, not as that of the nominal system without parameter uncertainties. This procedure is carried out by the trial and error adjustment of the matrices $Q \geq 0$ and $R > 0$ in (7) such that the following model:

$$\begin{aligned} \dot{x}^*(t) &= (A + D\Delta^*(t)E + BK_B)x^*(t) \\ &= (A + D\Delta^*(t)E - BR^{-1}B^TP)x^*(t) \end{aligned} \tag{15}$$

$$x^*(0) = x_0$$

$$\Delta^*(t) = \begin{cases} \frac{1}{\|D^TPx^*\|\|Ex^*\|} D^TPx^*x^{*T}E^T \\ \qquad \text{if } D^TPx^* \neq 0 \text{ and } Ex^* \neq 0 \\ \Delta^*(t^-) \qquad \text{otherwise} \end{cases}$$

exhibits a desired response, where the matrix P is the positive definite solution to (7). This design procedure is analogous to that of the standard LQ regulator design without parameter uncertainties, in which weighting matrices of quadratic cost function are adjusted so as to achieve a desired transient time-response. Of course, $x^*(t)$ is a sub-optimal response in the sense that exactly the same trajectory as the optimal time-response for the nominal system (A, B) might not be achieved.

Next, a nonlinear modification term is determined so as to minimize the error between the model

$$\dot{\widehat{x}}(t) = (A + D\widehat{\Delta}(t)E + BK_B)\widehat{x}(t),\ \ \widehat{x}(0) = x_0 \tag{16}$$

and

$$\dot{x}(t) = (A + D\Delta(t)E + BK_B)x(t) + Bv(t) \tag{17}$$

The error equation between (16) and (17) is given by

$$\begin{aligned} \dot{e}(t) &= (A + D\Delta(t)E + BK_B)e(t) \\ &\quad + D\left(\Delta(t) - \widehat{\Delta}(t)\right)E\widehat{x}(t) + Bv(t) \end{aligned} \tag{18}$$

Here, on the basis of the technique derived in [2], we define the additional input v as

$$v(t) = \begin{cases} -\frac{\alpha(e,\widehat{x})}{\|B^TPe\|^2} B^TPe(t) & \text{if } \|B^TPe\| \geq \delta \\ -\frac{\alpha(e,\widehat{x})}{\delta^2} B^TPe(t) & \text{otherwise} \end{cases} \tag{19}$$

$$\alpha(e, \widehat{x}) = \|D^TPe\| \cdot \|E\widehat{x}\| - e^TPD\widehat{\Delta}E\widehat{x}$$

where δ is a sufficiently small positive constant. In the space where $\|B^TPe\| \geq \delta$,

$$\frac{d}{dt}(e^TPe) \leq -\lambda_{\min}(Q + K_B^TRK_B)\|e\|^2$$

holds, and in the other space i.e.$||B^T Pe|| < \delta$,

$$\frac{d}{dt}(e^T Pe) \leq -\lambda_{\min}(Q + K_B^T R K_B)||e||^2 + \left(1 - \frac{||B^T Pe||^2}{\delta^2}\right) \alpha(e, \widehat{x})$$

holds, where $\alpha(e, \widehat{x}) \to 0$ as $\widehat{x} \to 0$. The minimization of the error means reducing cautiousness of the linear robust control $K_B x$, which also means that the model (16) exhibits a satisfactory transient behaviour close to the desired time-response given by (15). Nonlinear stabilizing control law such as (9) is not necessarily new. The main feature of the proposed method differs from previous work is that "active robust" controller for improving transient behaviour is established by combining appropriately designed linear and nonlinear "passive robust" control laws. The configuration of the proposed control system is shown in Figure 1.

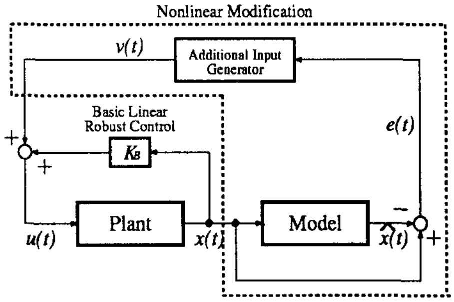
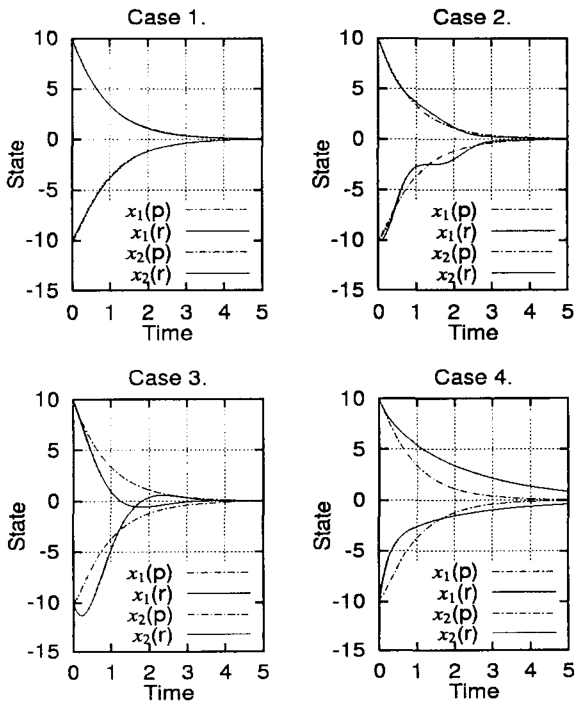


Fig.1 The configuration of the robust control system with nonlinear modification.

Example 2: The following figures indicate that the proposed control(dashed-dotted) improves the transient response by the robust control(solid) in the example 1 in a less cautious manner.



The simulation is carried out using the matrices

$$K_B = (-5.1656, -9.416), \ P = \begin{pmatrix} 43.149 & 5.165 \\ 5.165 & 9.416 \end{pmatrix}$$

which is the solution to (7) with $Q = 5I$.

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