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A Synthesis for Robust Servo System Based on Mixed H_2/H_∞ Control

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Abstract:

The purpose of this paper is to propose an approach to design a robust servo controller based on the Mixed H_2/H_∞ theory. In order to do this, we first modify the generalized plant for the usual H_∞ servo problem to a structure of the Mixed H_2/H_∞ minimization problem by virtue of the internal model principle. By doing this, we can divide specifications adopted for robust servo system design into H_2 and H_∞ performance criteria, respectively. Then, the mixed H_2/H_∞ problem is solved in order to find the best solution, by which we can minimize H_2-norm of the transfer function under the condition of H_∞-norm value, through Linear Matrix Equality (LMI).

1.Introduction

For a servo system design, the following three specifications are of practical interests: (1) internal stability of the closed-loop system which must be guaranteed; (2) desired feedback characteristics such as robust stability, sensitivity reduction and disturbance attenuation; (3) desired transient and steady-state properties such as robust tracking to reference inputs.

The H_∞ Control is a suitable technique to achieve the first two specifications, because they can be naturally expressed as H_∞ norm constraints. However, since the H_∞ Control is based on the maximum singular value of the transfer function matrix from disturbance to evaluation signals, it is inevitable that the response is rather conservative. Therefore, it is required to alleviate this phenomenon in order to meet the third specification. Recently, it has been proved that, by introducing H_2 specification into the H_∞ design, we could benefit from the H_2 and H_∞ control design simultaneously [1]. This approach is called the mixed H_2/H_∞ control, and the designer can determine the trade-off between H_2 (e.g. noise rejection) and H_∞ (e.g. robust stability) characteristics through this.

The purpose of this paper is to propose an approach to design a robust servo controller based on the mixed H_2/H_∞ control. The design objectives such as minimal tracking error and robust performance are first defined in terms of H_2 and H_∞ minimization. These objectives are then converted into linear matrix inequalities (LMIs). That is, the robustness stability criterion is expressed in an H_∞-norm LMI and the tracking performance is expressed in an H_2-norm LMI.

2. Problem Formulation

2.1 H_∞ Servo Problem

Consider a unity feedback control system shown in Fig.1, where *G(s)* and *K(s)* denote the plant and controller, respectively. Only finite dimensional linear time-invariant (LTI) systems and controller will be considered in this paper. Furthermore, for simplicity, all of the signals are regarded as scalar (SISO). We assume that the class of a reference signal is described as

$$r = G_R(s)r_o \tag{1}$$

where $G_R(s)$ is a transfer function which does not have stable poles, and r_o is an unknown constant vector. For example, in the case of step and ramp-type reference, $G_R(s)$ becomes $(1/s)$ and $(1/s^2)$, and r_o represents the magnitude of step and the slant of ramp signal. The purpose of the servo system design is to find a feedback compensator $K(s)$ that satisfies the following three specifications: 1) closed-loop internal stability, 2) desired feedback characteristics such as robust stability, sensitivity reduction and disturbance attenuation, and 3) robust tracking. Here, the robust tracking means that the output of the plant $y(t)$ tracks for any type of reference signal $r(t)$ without steady-state error, i.e., $\lim_{t\to\infty} e(t) = 0$, under the plant perturbation and/or step type disturbance inputs $d(t)$.

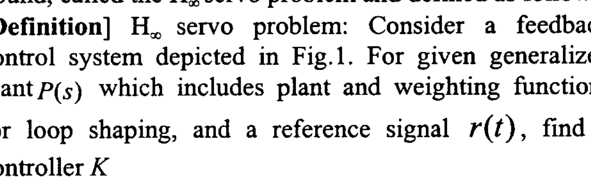


Fig.1 Feedback Control System

First, we consider the robust servo problem with H_∞ norm bound, called the H_∞ servo problem and defined as follows.

[Definition] H_∞ servo problem: Consider a feedback control system depicted in Fig.1. For given generalized plant $P(s)$ which includes plant and weighting functions for loop shaping, and a reference signal $r(t)$, find a controller *K*

$$u = Ky$$

satisfying the following three specifications:

(S1) *K* internally stabilizes *P*

(S2) $\|T_{ed}(s)\|_\infty < \gamma$

(S3) *K* achieves the robust tracking property for the reference signal, where $T_{ed}(s)$ represents the transfer function from d to e.

There have been much research related to the H_∞ servo problem. Sugie et al.[2] derived sufficient and necessary conditions for the existence of a solution K of the H_∞ servo problem.

[Theorem 1] Consider the feedback control system depicted in Fig.1. It is assumed that the plant $G = ND^{-1}$ and the reference generator $G_R = \tilde{D}_R^{-1}\tilde{N}_R$ are described by using left and right coprime factorization, respectively. Then, the necessary and sufficient condition for the existence of a solution K to the H_∞ servo problem is that the following three conditions are satisfied simultaneously.

(C1) $K \in \Omega(G)$, where Ω represents the set of stabilized compensators to plant G.

(C2) $D_K/\alpha_R \in RH_\infty$, where α_R denotes the largest invariant factor of $\tilde{D}_R$, D_k is a denominator when it is described by right coprime factorization and RH_∞ is the set of all stable real rational functions.

Hence, H_∞ servo controller must have the structure as depicted in Fig.2, where K_r internally stabilizes G/α_R.
In the case of step and sinusoid-type reference signals, α_R becomes

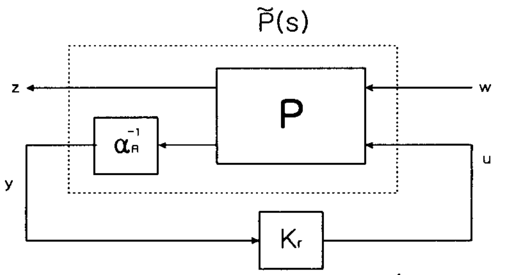


Fig. 2 The Original Problem[1]

$$\alpha_R = \begin{cases} \dfrac{s}{s+\beta} \text{ (step)} \\ \dfrac{s^2+\omega_r^2}{(s+\beta)^2} \text{ (sinusoid)} \end{cases} \tag{2}$$

where ω_r is the frequency of a reference signal and β is an arbitrary constant.

2.2 Mixed H_2/H_∞ Optimal Design Problem

The basic block diagram used in this paper is given in Fig.3, in which the generalized plant P is given by the state-space equations

$$P: \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1 w + \mathbf{B}_2 u \\ z_\infty = \mathbf{C}_\infty \mathbf{x} + d_{\infty 1} w + d_{\infty 2} u \\ z_2 = \mathbf{C}_2 \mathbf{x} + d_{21} w + d_{22} u \\ y = \mathbf{C}_y \mathbf{x} + d_{y1} w \end{cases} \tag{3}$$

where $\mathbf{x} \in R^n$ is state vector, u is the control input, w is an exogenous inputs (such as disturbance signals, sensor noise), y is the measured output and $\mathbf{z} = [z_\infty \; z_2]^T$ is a vector of output signal related to the performance of the control system (z_∞ is related to the H_∞ performance and z_2 is related to the H_2 performance).

Let T_{zw} be the closed transfer function from w to z for the system P connected with output-feedback control law $u = K_r y$. Our goal is to compute a dynamical output feedback controller K_r that meets H_2 and H_∞ performance on the closed-loop behavior.

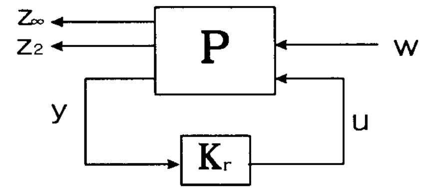


Fig.3 Block diagram of Mixed H_2/H_∞ control

$$K_r: \begin{cases} \dot{\mathbf{x}}_K = \mathbf{A}_K \mathbf{x}_K + \mathbf{B}_K y \\ u = \mathbf{C}_K \mathbf{x}_K + d_K y \end{cases} \tag{4}$$

The closed-loop system T_{zw} has the following description

$$T_{zw}: \begin{cases} \dot{\mathbf{x}}_{cl} = \mathbf{A}_{cl} \mathbf{x}_{cl} + \mathbf{B}_{cl} w \\ z_\infty = \mathbf{C}_{cl1} \mathbf{x}_{cl} + d_{cl1} w \\ z_2 = \mathbf{C}_{cl2} \mathbf{x}_{cl} + d_{cl2} w \end{cases} \tag{5}$$

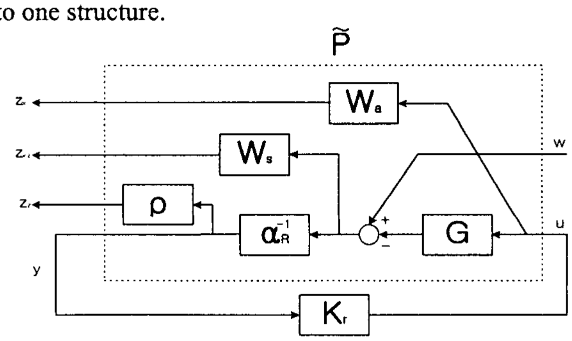
The problem we concerned with can be summarized as minimizing the H_2 norm of the channel $w \to z_2(:T_2)$, while keeping the bound γ on the H_∞ norm of the channel $w \to z_\infty(:T_\infty)$, i.e.

$$\min \|T_2\|_2 \text{ subject to : } \|T_\infty\|_\infty < \gamma$$

Since this problem can be reformulated as a convex optimization problem [1], the optimal solution under the given value of γ can be obtained through LMI. As efficient interior-point algorithms are now available to solve the generic LMI problems, the mixed H_2/H_∞ problem is solved without much difficulty in order to find the best trade-off between the H_2 and H_∞ minimization.

3. Main results

In order to find a robust servo controller $K(s)$ which satisfies desired feedback properties (H_∞ performance) and good reference response (H_2 performance) simultaneously, we have to adopt the mixed H_2/H_∞ control system rather than the conventional H_∞ control theory. That is, the control objectives are divided into H_∞ and H_2 performance criteria. Therefore, it is necessary to combine two criteria into one structure.



[1] Of course, the meaning of the signal y in Fig.1 and Fig.2 is different. In Fig. 2, z represents the output of interest and w is an exogenous input.

Fig. 4 The proposed generalized plant for Mixed H_2/H_∞ control

A. H_∞ Control Problem

In Fig.4, $W_a(s)$ is an weighting function related to the plant uncertainty (additive uncertainty) and $W_s(s)$ represents weighting for the sensitivity function. Therefore, we can summarize robust tracking H_∞ control problem as follows:

(S1) $K_r(s)$ stabilizes $\tilde{P}(s)$.

(S2) $\|T_\infty(s)\|_\infty = \left\| \begin{matrix} T_{z_{\infty 1}w}(s) \\ T_{z_{\infty 2}w}(s) \end{matrix} \right\|_\infty < \gamma$

where $T_{z_{\infty 1}w}(s)$ denotes the transfer function from w to $z_{\infty 1}$ and it is related to robust stability requirement (for additive uncertainty)

$$\|T_{z_{\infty 1}w}\|_\infty = \|(1+GK)^{-1}KW_a\|_\infty < \gamma \quad (6)$$

where $K = \alpha_R^{-1}K_r$

The nominal performance condition is reflected by $\|T_{z_{\infty 2}w}(s)\|_\infty < \gamma$, where $T_{z_{\infty 2}w}(s)$ denotes the transfer function from w to $z_{\infty 2}$. In this case, if (S2) is satisfied under the condition of γ=0.5, then robust performance for the given plant can be guaranteed outright, because (S2) will satisfy the SISO robust performance test for additive uncertainty given by [3]

$$\left\| |T_{z_{\infty 1}}(s)| + |T_{z_{\infty 2}}(s)| \right\|_\infty < 1 \quad (7)$$

The robust performance that was given in (7) is necessary and sufficient, and the left-hand side is actually the peak value of μ. In section 4, we will check and analyze this value of the closed-loop system, when the controller $K(s)$ is determined by the mixed H_2/H_∞ and μ control algorithm, respectively.

By virtue of the Bound Real Lemma, the H_∞ norm of $T_\infty(s)$ is smaller than γ if and only if there exists a symmetric positive definite matrix $\mathbf{X}_\infty$ with

$$\begin{pmatrix} \mathbf{A}_{cl}^T\mathbf{X}_\infty + \mathbf{X}_\infty\mathbf{A}_{cl} & \mathbf{X}_\infty\mathbf{B}_{cl} & \mathbf{C}_{cl\infty}^T \\ \mathbf{B}_{cl}^T\mathbf{X}_\infty & -\gamma & d_{cl\infty} \\ \mathbf{C}_{cl\infty} & d_{cl\infty} & -\gamma \end{pmatrix} < 0 \quad \text{(8-a), (8-b)}$$

$$\mathbf{X}_\infty > 0$$

where all the matrices $\mathbf{A}_{cl}, \mathbf{B}_{cl}, \mathbf{C}_{cl\infty}$ and $d_{cl\infty}$ are defined in (4).

B. H_2 control Problem

In order to obtain the desired reference response, we must utilize the H_2 control theory. That is, the H_2 norm minimization of the transfer function $T_{z_2w}(s)$ from w to z_2 in Fig.5 is considered, where ρ is a weighting constant.

It is well known that this norm can be computed as $\|T_{z_2w}\|_2^2 = tr(\mathbf{C}_{cl2}\mathbf{W}_o\mathbf{C}_{cl2}^T)$, where W_o solves the Lyapunov equation

$$\mathbf{A}_{cl}\mathbf{W}_o + \mathbf{W}_o\mathbf{A}_{cl}^T + \mathbf{B}_{cl}\mathbf{B}_{cl}^T = \mathbf{0} \quad (9)$$

Since $\mathbf{W}_o < \mathbf{W}$ for any $\mathbf{W}$ satisfying

$$\mathbf{A}_{cl}\mathbf{W} + \mathbf{W}\mathbf{A}_{cl}^T + \mathbf{B}_{cl}\mathbf{B}_{cl}^T < \mathbf{0} \quad (10)$$

It is readily verified that $\|T_{z_2w}\|_2^2 < \nu$ if and only if there exists W >0 satisfying (10) and $tr(\mathbf{C}_{cl2}\mathbf{W}\mathbf{C}_{cl2}^T) < \nu$ [1]. With auxiliary parameter Q, the following analysis result has been known:

[Theorem 2] $\mathbf{A}_{cl}$ is stable and $\|T_{z_2w}\|_2^2 < \nu$ if and only if there exist symmetric $\mathbf{X}_2 = \mathbf{W}^{-1}$ and Q such that

$$\begin{pmatrix} \mathbf{A}_{cl}^T\mathbf{X}_2 + \mathbf{X}_2\mathbf{A}_{cl} & \mathbf{X}_2\mathbf{B}_{cl} \\ \mathbf{B}_{cl}^T\mathbf{X}_2 & -\mathbf{I} \end{pmatrix} < 0 \quad \text{(11-a)}$$

$$\begin{pmatrix} \mathbf{X}_2 & \mathbf{C}_{cl2}^T \\ \mathbf{C}_{cl2} & \mathbf{Q} \end{pmatrix} > 0 \quad \text{(11-b)}$$

$$tr(\mathbf{Q}) < \nu \quad \text{(11-c)}$$

C. Mixed H_2/H_∞ Control

The Mixed H_2/H_∞ controller K_r has to satisfy both of the following criteria simultaneously

$$\|T_{z_\infty w}\|_\infty < \gamma \quad (12)$$

$$\|T_{z_2w}\|_2 < \nu \quad (13)$$

In order to keep the tractability of the constrained optimization problem, the following assumption is considered [1].

$$\mathbf{X}_\infty = \mathbf{X}_2 \equiv \mathbf{P} \quad (14)$$

Therefore, notice that X can be written as

$$\mathbf{P} = \begin{bmatrix} \mathbf{X} & * \\ * & * \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & * \\ * & * \end{bmatrix}^{-1} \quad (15)$$

where dim(X)=dim(Y)=dim(A_{cl}) and $\begin{bmatrix} \mathbf{X} & \mathbf{I} \\ \mathbf{I} & \mathbf{Y} \end{bmatrix} > 0$ is coupling LMI. The solution X, Y, and Q under the constraints of (12),(13) is dependent on the value of γ and ν, and are obtained using any available software such as Matlab LMI toolbox [4].

4. Design Examples and Analysis

In this section, we'll show a example in order to evaluate the effectiveness of the proposed structure for robust servo system.

4.1 Example 1(F-18 Aircraft)

In [5], the simplified model of a F-18 Aircraft was composed of three parts: nominal model $G_g(s)$, actuator model $G_\delta(s)$ and time-delay model $G_d(s)$, which were given as follows:

$$G_g(s) \equiv \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} -1.994 & 0.9853 & -0.2852 \\ -19.44 & -1.427 & -33.44 \\ 54.35 & 0.469 & 7.774 \end{bmatrix} \quad (16)$$

$$G_\delta(s) = \begin{bmatrix} -40 & 1 \\ 40 & 0 \end{bmatrix} \quad G_d(s) = \begin{bmatrix} -29.85 & 1 \\ 59.70 & -1 \end{bmatrix}$$

It was desired to have the controller provide robust performance at an off-nominal design point. The perturbed system was given below under the conditions of the perturbed point

$$G_p(s) = \begin{bmatrix} -2.328 & 0.9831 & -0.3010 \\ -30.44 & -1.493 & -39.43 \\ 67.42 & 0.490 & 8.723 \end{bmatrix} \quad (17)$$

First, after designing a robust controller by using the proposed structure of the previous section, we analyze its robust performance related to H_2 and/or H_∞ properties, and compare the results with other method such as H_2, H_∞ and μ synthesis control algorithms.

We showed a generalized plant for the mixed H_2/H_∞ control in Fig.5, where $G_{nom}(s)$ represents a system including a nominal, actuator and time-delay model. $W_s(s)$ and $W_r(s)$ defined in (18) mean the sensitivity weighting function and the additive uncertainty of the nominal plant, respectively.

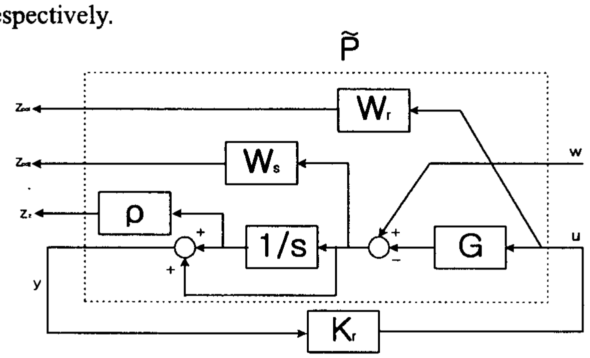
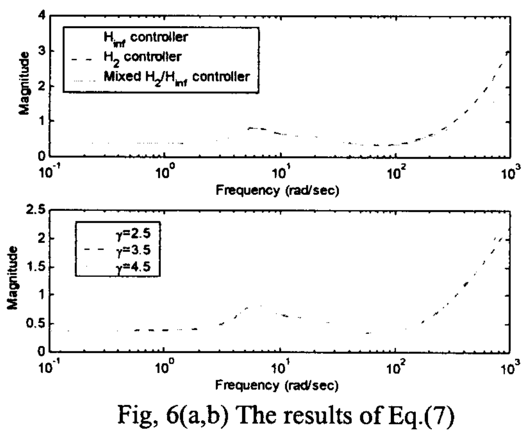


Fig.5 The generalized plant for Example 1

$$W_s(s) = \frac{0.2(s+10.001)}{s+0.001}, \quad W_r(s) := G_p(s) - G_g(s) \qquad (18)$$

To investigate the robust performance of the mixed H_2/H_∞ controller, we, first, check the value of (7). From Fig. 6(a), we know that most controller meet the robust performance condition of (7) under the frequency region of 300 rad/sec. Fig. 6(b) implies that increasing γ results in weak stability robustness in high frequencies.



Fig, 6(a,b) The results of Eq.(7)

In Fig.7, we show the results of step responses for H_2, H_∞, μ designs and a mixed H_2/H_∞ design with γ=2.5, respectively. One can see that better transient responses are obtained by H_2 and the mixed H_2/H_∞ design. In the case of H_2 design, however, poor robust stability margin is unavoidable ($H_2 \rightarrow 0.0071$, mixed $H_2/H_\infty \rightarrow 0.819$).

The step responses with respect to the perturbed plant $G_p(s)$ is shown in Fig.8. We know that the output tracks the reference input very well in spite of parameter variations. Furthermore, we confirmed that, if we increase the value of γ (that is, loose H_∞ characteristic), the settling time is shorter (tighten H_2 characteristic) as expected.

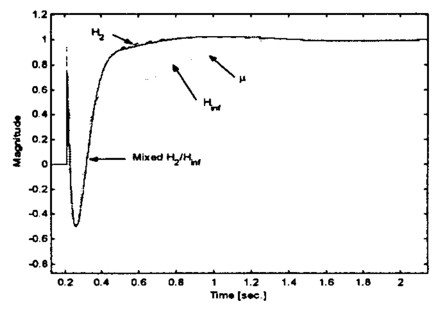


Fig.7 The results of Step Responses (Nominal Plant)

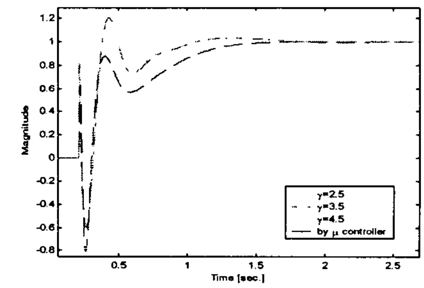


Fig.8 The Results of Step Responses (Perturbed Plant)

5. Conclusion

In this paper, we have proposed a generalized plant structure for the mixed H_2/H_∞ control theory such that the resultant closed-loop system achieves not only robust performance but also good reference response, simultaneously. That is, the design objectives such as better transient response and robust tracking were defined in terms of H_2 and H_∞ optimization theory, then these objectives were converted into linear matrix inequalities. Efficient interior point algorithms have been developed to solve this optimization problem numerically.

From the results of examples, it has been found that the mixed H_2/H_∞ controller provides better tracking performance than other methods under the parameter variations. And we verified that a robust controller could be designed easily by adjusting the value of γ which represents the trade-off between H_2 and H_∞ characteristic.

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