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Study of New Control Method for Linear Periodic System

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ABSTRACT

The purpose of this study is to provide the new method for selection of a close to optimal scalar control of linear time-periodic system. The case of scalar control is considered, the gain matrix being assumed to be at worst periodic with the system period T. The form of gain matrix may have various kinds but must have same period, for example, one of each element being represented by Fourier series. As the optimal gain matrix I consider the matrix ensuring the minimum value of the larger real part of the Poincare exponents of the system. Finally we present a pole placement algorithm to make the given system be stable. It is possible to determine the stability of the given periodic system without get the analytic solution. The application of the method does not require the construction of the Floquet solution. At present state of determination of the gain matrix for this case will be done only by systematic numerical search procedures.

1. Introduction

Since Floquet theory which is the conventional method and used widely in the study of linear time periodic systems , especially in the field of celestial mechanics. The usual use of Floquet theory is in determining the stability of a known periodic orbit. And the study about the control of unstable periodic orbit was considered by Wiesel, W. and Shelton, W.[1] They applied Floquet theory to the problem of designing a control system for a satellite in an unstable periodic orbit. Stationkeeping near an unstable periodic orbit has also been studied by Breakwell, J.V., Kamel, A. and Ratner, M.J.[2] A method for selection of a scalar control of two dimensional linear time periodic system has been proposed by Radziszewski and Zaleski[3] and utilizes the quadratic form of Lyapunov function in the motion equations case having varying coefficients. The system under consideration may be assigned either in closed-loop state or in modal variables as in Calico, R.A. and Wiesel, W.E.[4]. In this paper, the linear systems which have the time period or certain time interval should be necessarily studied in the sense of dynamic stability and design method of controller to stabilize the original unstable systems. The time periodic systems are easily found in scientific and industrial fields. For example, the same motion and movements during time interval are included as a periodic systems in the large sense of concept. The Lyapunov stability criteria to satisfy the energy convergence is basically used and new technique of pole placement is derived along the whole time period. The proposition of this research is the derivation of new criteria of the stability and technique to design of the controller to make the system be stable. The final acquisition of this study is the numerical procedures to decide the system's over all stability and find the coefficients of the controller(gain matrix) to satisfy the new stability criteria. The selected form of Lyapunov function (satisfying the Lyapunov direct method) is quadratic function and numerical programs will make the real part of characteristic root of system equation be minus value. That's why the optimization method should be applied to this research. The method, based on two step optimization procedure, allow to find the approximate optimal gain matrix. The various form of controller can be the candidate but the Fourier series is utilized in this research.

2. Analysis

(1) System Analysis

The general form of autonomous system is $\dot{x}=f(x)$ and the time periodic state vector is $x(t)=x(t+T)$. The small deviation of the variable will be expressed as $\delta x=x(t)-x^{*}(t)$, and the linearized equation is $\frac{d}{dt}(\dot{x}^*) = \frac{\partial f}{\partial x}[x(t)]\dot{x}^*$ where $x^*(t) = x^*(t+T)$.

The time-periodic system under consideration is of the form

$\dot{x} = P(t)x$ *where* $x(t_0) = x_0$, $x \in R^n$, $t \in (t_0, \infty)$ (1)

where *P(t)* is periodic with the periodic with the period *T*. The state transition matrix Φ has the relationships :

$$\frac{d\Phi}{dt} = P(t)\Phi \quad \Phi(t+T, t_0) = \Phi(t, t_0)C$$

$$\frac{d\Phi(t+T, t_0)}{dt} = \frac{d\Phi(t+t_0, t_0)}{dt}C$$

$$\frac{d\Phi(t+T, t_0)}{dt} = P(t)\Phi(t, t_0)C = P(t)\Phi(t+T, t_0)$$

where $C = e^{RT}$

$$\Phi(t, t_0) = \Psi(t, t_0)e^{(t-t_0)}$$

$$\Rightarrow \Psi(t, t_0) = \Psi(t+T, t_0)$$

$$\Rightarrow \Psi(t+T, t_0) = \Phi(t+T, t_0)e^{-R(t+T-t_0)}$$

$$\Rightarrow \Psi(t+T, t_0) = \Phi(t, t_0)e^{-R(t-t_0)} = \Psi(t, t_0) = I$$

$$\Rightarrow \Phi(t_0+T, t_0) = \Psi(t_0+T, t_0)e^{RT} = e^{RT}$$

Consequently, the state transition matrix and solution to (1) consistes of a periodically modulated exponential matrix function. Therefore, (1) is asymptotically stable if the eigenvalue of R all have negative real parts. That is

$$\begin{aligned} &det[I\rho - R] = 0 \quad where\ Re(\rho_i) < 0\ ,\ i = 1, 2, , \\ &det[\lambda I - e^{RT}] = 0 \quad where\ |\lambda_i| < 1\ ,\ i = 1, 2, , \end{aligned} \quad (2)$$

Here ρ's are called characteristic exponents and λ's are characteristic multipliers of matrix A(t). They have the relationship as :

$$\rho_i = \frac{1}{T}\ln\lambda_i \quad (3)$$

These two characteristic numbers are very important elements to decide the system's stability.

The closed loop state equation of the controlled system is of the form

$$\dot{x} = [A(t) + G(t)K(t)]x \quad (4)$$

where 1×n matrix *G(t)* describes the distribution of the control in the system n×1 matrix *K(t)* is the gain matrix (both matrices are assumed to be periodic with the same period *T*). We assume the elements of the gain matrix to be represented by truncated Fourier series,

$$G(t) = [g_1(t)\ \ g_2(t)\ - \ -]^T \quad (5)$$

$$K(t) = [k_1(t)\ \ k_2(t)\ - \ -] \quad (6)$$

$$k_i(t) = k_i(0) + \sum_{n=0}^{m}[\ Ak_i(n)\ \sin\frac{2\pi nt}{T} + Bk_i(n)\ \cos\frac{2\pi nt}{T}] \quad (7)$$

Until these introductory remark will be utilized and compared with new criteria which is set with the Lyapunov stability concept.

(2) Lyapunov function in Quadratic form

The Lyapunov stability concept basically utilizes the energy conrgence of the system. The total energy of the system is definitely positive and if the rate of this energy with time is decreasing, it should be told stable system, Using this conditions, the certain poitive function(called Lyapunov candidate function) is chosen first, and the negativeness of the time derivcative of this function should be checked secondly. In this paper, the Lyapunov candidate function is selected as the quadratic form.

Introduce the quadratic form of Lyapunov function

$$V(x) = x^T Q x \quad (8)$$

where Q is the constant and symmetric, positive definite. Introduce also the generalized norm $||x||_s$ induced by the scalar product $x^T Q x$. The Lyapunov derivative of (7) along solution of (1) is of the form

$$\begin{aligned} \frac{dV(t,x)}{dt} &= \dot{x}^T Q x + x^T Q \dot{x} = (Px)^T Q x + x^T Q(Px) \\ &= x^T A^T Q x + x^T Q P x = x^T(P^T Q + QP)x \end{aligned} \quad (9)$$

Multiplying Q^{-1} to the $P^T Q + QP$ matrix, also introduce now the auxiliary matrix :

$$C(t) = Q^{-1}P^T Q + P \quad (10)$$

Due to the periodicity of *P(t)* the *C(t)* is periodic too. The eigenvalues of the matrix C(t) are the same as these of the symmetric matrix $(P^T Q + QP)$ and thus they are real.

Denoting by $\Lambda_{max}(t)$ (also periodic) the maximal eigenvalue of the matrix C(t), the estimation holds

$$V(x(t)) \le V(x_0)\exp[\int_0^T \Lambda_{max}(t)\,dt] \quad (11)$$

(3) Stability criteria

In the previous section, the linear quadratic Lyapunov function is expressed by matrix calculation. And the time derivative of this function is also calculated simply with the linear algebra. This is of the form as:

$$\frac{\dot{V}}{V} = \frac{x^T(P^TQ+QP)x}{x^TQx} \quad (12)$$

Using the linear algebra calculation, the results are

$$\frac{\dot{V}}{V} = \frac{dV/dt}{V} = x^T(Q^{-1}P^TQ+P)x = \Lambda(t) \quad (13)$$

$$\Lambda_{\min}[Q^{-1}P^TQ+P] < \Lambda(t) < \Lambda_{\max}[Q^{-1}P^TQ+P] \quad (14)$$

$$\frac{dV}{V} = \Lambda(t)\,dt \quad (15)$$

$$V(x(t)) = V(x_0)\exp[\int_{t_0}^{t}\Lambda(t)\,dt] \quad (16)$$

$$\lim_{t\to\infty} V[x(t)] = V[x(0)]\,e^{[\lim_{t\to\infty}\frac{1}{t-t_0}\int_{t_0}^{t}\Lambda(t)\,dt\,(t-t_0)]} \quad (17)$$

In this result, matrix $C(t)$ and the eigenvalue of this matrix is the main factor in determining the stability of system. The relationship between periodic system equation and the Lyapunov function(energy function) should be examined by the Lyapunov direct method and this will be the main procedures and results. And the final step is to find the controller to make the original system be stable with leaving the period.

Replacing $t=t_0+nT$,

$$\frac{1}{t-t_0}\int_{t_0}^{t}\Lambda_{\max}(t)\,dt = \frac{1}{t_0+nT}\int_{t_0}^{t_0+nT}\Lambda_{\max}(t)\,dt = \frac{1}{T}\int_{0}^{T}\Lambda_{\max}(t)\,dt \quad (18)$$

This equation reveals that if (18) is negative then V(x(t)) is bounded and converges to zero as time goes to infinity. The time periodic system (1) is said to be asymptotically stable if the following condition holds.

$$\frac{1}{t-t_0}\int_{t_0}^{t}\Lambda_{\max}(t)\,dt = \frac{1}{T}\int_{0}^{T}\Lambda_{\max}(t)\,dt < -\varepsilon \quad (18)$$

for some $\varepsilon > 0$.

Denoting $\Lambda = \frac{1}{T}\int_{0}^{T}\Lambda_{\max}(t)\,dt$, I get the estimation of the induced norm at the solution

$$||x(T)||_s \le ||x(0)||_s \exp||\Lambda|| . \quad (20)$$

(4) Main Results

The maximum eigenvalue of the matrix C(t) is the function of many variables including all state variables and time, matrices K and Q(quadratic Lyapunov function). Let's try to clear this relationship between $\Lambda_{\max}$ and related variables with second order case. The maximum eigenvalue of the matrix C(t) is of the form

$$\Lambda_{\max} = \frac{Tr(C)}{2} + \sqrt{(\frac{Tr(C)}{2})^2 + \det C} . \quad (21)$$

By direct computation one can see that the trace TrC of the matrix C(t) depemds only on the coefficients of the Fourier series, while the determinant detC depends on these coefficients as on the elements of the matrix $Q(q_{11}, q_{12}, q_{21}, q_{22})$

$$TrC = TrC(t, k_i(0), Ak_{i(1)},\ldots, Ak_i(m), Bk_i(1),,,, Bk_i(m)),\ i=1,2 \quad (22)$$

$$detC = detC(t, k_i(0), Ak_i(1),, Ak_i(m), Bk_i(1),, Bk_i(m), q_{11}, q_{12}, q_{21}, q_{22}),\ i=1,2 \quad (23)$$

where $k_i(t) = k_i(0) + \sum_{n=0}^{m}[Ak_i(n)\sin\frac{2\pi nt}{T} + Bk_i(n)\cos\frac{2\pi nt}{T}]$

This fact implies the two step procedures of optimization of Λ. At the first sub step the quantity Λ is minimized while varying the values of the elements of the matrix Q and freezing the values of the Fourier series coefficients. At the second sub step Λ is minimized while varying the values of the Fourier series coefficients and freezing the previously chosen values of the elements of the matrix Q. This procedure is continued till the improvement of the value of Λ becomes negligibly small. The simplification of the problem may be achieved by using the Cauchy-Schwartz inequality to approximate the integral in the expression for Λ.

In this research, the first selection of Lyapunov candid function will not satisfy Lyapunov stability conditions. Two possible representations of the matrix Q were considered. The first one utilized the relation $Q=M^TDM$ where M is an orthogonal matrix, D is diagonal one, such that the element $d_{11}=1$ and $d_{22}>0$. For the second representation

$$Q = \begin{bmatrix} \alpha & \beta \\ \beta & 1 \end{bmatrix} \quad (24)$$

the positive definiteness condition is given directly $\alpha - \beta^2 > 0$.

The form of the matrix Q was selected for programming. For the first sub step the program requires the constrained minimization method. The Box's Complex Method[16] was selected for this purpose. For the second sub step the Simplex Method[17] was applied.

3. Applications

The system of the form (1) was selected with the goal to determine the gain matrix minimizing the larger real part of the Poincare exponent.

Example 1.

The system equation is of the form:

$$\dot{x} = Ax = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix} x$$

$$G(t) = [\sin t \quad -\sin t]$$

$$k_i(t) = k_i(0) + \sum_{n=0}^{m} [\, Ak_i(n) \sin\frac{2\pi nt}{T} + Bk_i(n) \cos\frac{2\pi nt}{T}] \quad , \quad i = 1, 2$$

The initial gain matrix was chosen as having all the coefficients equal to zero.

For these values $\Lambda = +0.1$ which shows instability of the system.

Table 1. summarizes the values of gains from the two step optimization procedure for Example 1. The resulting Poincare exponent estimate attains the value $\Lambda = -0.10999806 < 0$ showing that the system is stabilized.

i	1	2
$k_i(0)$	0.0918	0.2195
$Ak_i(1)$	-1.0440	0.0129
$Bk_i(1)$	0.0475	0.4887
$Ak_i(2)$	0.1236	-0.0049
$Bk_i(2)$	0.4224	0.4305
$Ak_i(3)$	0.0562	-0.0138
$Bk_i(3)$	0.3383	0.4781
$Ak_i(4)$	0.1055	-0.0047
$Bk_i(4)$	0.2163	0.4349
$Ak_i(5)$	0.7579	0.2270
$Bk_i(5)$	0.0411	0.2440

Table 1. Output Data for Example 1

Example 2

The purpose of this example is to prove the performance of the numerical procedure. The system quoted after [ ] is of the form (1).

$$\dot{x} = P(t)x = \begin{bmatrix} -1 + a\cos^2 t & 1 - a\sin t\cos t \\ -1 - a\sin t\cos t & -1 + a\sin^2 t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A closed form solution of the system is known and is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{(a-1)t}\cos t & e^{-1}\sin t \\ -e^{(a-1)t}\sin t & e^{-1}\cos t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

for all values of a. The solution shows that asymptotic stability requires that $a < 1$ and that for all values of $a \leq 0$ the value of Λ may not be less than -1.

Rewriting the system in form (2) gives

$$\dot{x} = [A(t) + G(t)K(t)]x$$

$$P(t) = A(t) + G(t)K(t) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} [\, a\cos t \quad -a\sin t]$$

For selected series of initial values of $'a'$ as shown at the table 2 the results of procedure converge with the error of 0.09% to the expected value of $\Lambda = -1$.

Initial Value of $'a'$	Λ
-2	-0.9999996
-5	-0.9999994
0	-0.9998086
2	-0.9991928
5	-0.9993884

Table 2. Output Data for Example 2

4. Conclusions

The proposed simple procedure provides an effective method and tool for control of linear time periodic systems. Example 2 shows that if converges to the lower limit of the exact value of the system exponent.

The solutions of the control problem of two dimensional time periodic system will be provided in the form of the program calculating close to optimal control parameters. The method will enable selection of the close to optimal, scalar control of two modes resulting in simpler control system. It will be assumed that the Fourier series development of coefficients($K_i(t)$) may be truncated after its few(fifth in this paper) terms. Methods have been developed which allow the solution to broad range of interesting problems. Much work remains to be done, however, both in extending the theory to more general cases, and in giving experience with actual applications of the higher dimensional systems. Finally, analysis of the gain matrix will be provided estimating their influence about the objective time periodic systems. The conclusions of this project will give certain informations and guide lines for control of general periodic systems.

V. References

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