신뢰 $\mathcal{H}_\infty$ 제어의 선형 행렬 부등식 방법

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LMI Approach of Reliable $\mathcal{H}_\infty$ Control

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Abstract

This note addresses the problem of reliable $\mathcal{H}_\infty$ output-feedback control design for linear systems with actuator and/or sensor failures. An output feedback control design is proposed which stabilizes the plant and guarantees an $\mathcal{H}_\infty$-norm bound on attenuation of augmented disturbances including all admissible actuator/sensor failures. Based on the linear matrix inequality (LMI) approach, the output-feedback controller design method is constructed by formulating to LMIs that cover all failure cases. Effectiveness of this controller is validated via a numerical example.

I. Introduction

The robust and reliable control problems have been of high interest in the control engineering field. Recently, an attempt to tackle both problems has been suggested by Seo and Kim[4], who developed an ARE-based robust and reliable $\mathcal{H}_\infty$ control methodology via output-feedback for linear uncertain systems with parameter uncertainties and allowable actuator/sensor failures. However, a prior condition is required to solve the coupled nonlinear matrix equations for output-feedback control, which can obtain so hardly even if its solution exists. On the other hand, the LMI approach has a numerically tractable algorithm due to its convex optimization[3]. Thus, one may regard the LMI approach as an effective alternative to the ARE-based one. A reliable framework formulated in LMIs has been provided in [2] where the cross decomposition algorithm is needed for constructing dynamic output-feedback controller in view of $\mathcal{H}_2$ optimization.

In this paper, we present new design methods for robust and reliable $\mathcal{H}_\infty$ output-feedback control problems using a simple LMI formulation. Furthermore, our approach does not need to perform additive algorithm such as cross decomposition and deals with soft failure type as well as hard one.

II. Main Results

In this section, we design a dynamical output-feedback controller which can stabilize the linear system with $\mathcal{H}_\infty$-norm bound γ against any admissible actuator/sensor failures.

Consider the following continuous-time linear system, which is given by the state-space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_w w(t) \qquad (1) \\ z(t) &= C_z x(t) + D_z u(t) \\ y(t) &= Cx(t) + D_w w(t) \end{aligned}$$

Let an actuator failure $f_a \in \Omega_a$ be occurred. Then, the set of actuator which operates normally is $r_a \in \{1, 2, \cdots, m\} - f_a$, and B, D_z, and u can be decomposed, without loss of generality, by

$$B = (B_{r_a}\ B_{f_a}),\ \ D_z = (D_{zr_a}\ D_{zf_a}),\ \ u = (u_{r_a}^T\ u_{f_a}^T)^T$$

respectively. On the other hand, let an sensor failure $f_s \in \Omega_s$ be occurred at the same time. Then, the set of sensor which operates normally is $r_s \in \{1, 2, \cdots, o\} - f_s$, and also C, D_w, and y can be, without loss of generality, decomposed by

$$C = (C_{r_s}^T\ C_{f_s}^T)^T,\ \ D_w = (D_{wr_s}^T\ D_{wf_s}^T)^T,\ \ y = (y_{r_s}^T\ y_{f_s}^T)^T$$

respectively. In this framework, the system is represented by the following post-fault model.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_{r_a} u_{r_a}(t) + (B_w\ B_{f_a}) \begin{pmatrix} w(t) \\ u_{f_a}(t) \end{pmatrix} \\ z_{r_a}(t) &= C_z x(t) + D_{zr_a} u_{r_a}(t) \qquad (2) \\ y(t) &= \begin{pmatrix} C_{r_s} \\ C_{f_s} \end{pmatrix} x(t) + \begin{pmatrix} D_{wr_s} \\ D_{wf_s} \end{pmatrix} w(t) \end{aligned}$$

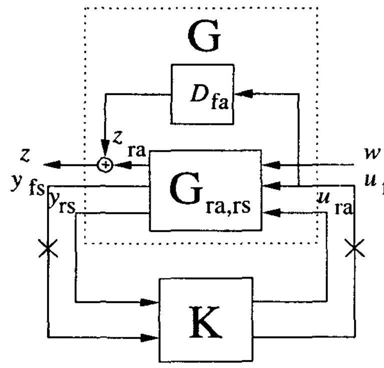

Figure 1: The output feedback control system with actuator/sensor failures

Under the decompositions and the assumption that $D_K = 0$, a dynamic output-feedback controller to meet $\mathcal{H}_\infty$ norm-bound on the closed-loop behavior is formulated as

$$\begin{aligned}\dot{\zeta}(t) &= A_K\zeta(t) + B_K y(t) \\ &= A_K\zeta(t) + B_{Kr_s}y_{r_s}(t) + B_{Kf_s}y_{f_s}(t) \quad (3) \\ u(t) &= C_K\zeta(t) \\ &= \begin{pmatrix} u_{r_a}(t) \\ u_{f_a}(t) \end{pmatrix} = \begin{pmatrix} C_{Kr_a} \\ C_{Kf_a} \end{pmatrix}\zeta(t)\end{aligned}$$

where A_K, B_K, and C_K are decomposed in appropriate dimensions.

Once the controller is applied to the system (2), the closed loop system T_{r_a,r_s} is represented by

$$T_{r_a,r_s}\begin{cases} \dot{x}_{cl}(t) = \mathcal{A}_{r_a,r_s}x_{cl}(t) + \mathcal{B}_{r_a,r_s}w_{f_a,f_s}(t) \\ z_{r_a}(t) = \mathcal{C}_{r_a}x_{cl}(t) + \mathcal{D}_{r_a,r_s}w_{f_a,f_s}(t) \end{cases} \quad (4)$$

where $x_{cl}(t) = \begin{pmatrix} x(t) \\ \zeta(t) \end{pmatrix} \in \mathcal{R}^{2n}$ is the state of the closed-loop system, $w_{f_a,f_s}(t) = \begin{pmatrix} w(t) \\ u_{f_a} \\ y_{f_s} \end{pmatrix}$ is the additive disturbance including the elements of actuator and sensor failures, $z_{r_a}(t) \in \mathcal{R}^p$ is the controlled output, and the closed-loop system matrices ($\mathcal{A}_{r_a,r_s}$, $\mathcal{B}_{r_a,r_s}$, $\mathcal{C}_{r_a}$, $\mathcal{D}_{r_a,r_s}$) are expressed as follows:

$$\left(\begin{array}{c|c} \mathcal{A}_{r_a,r_s} & \mathcal{B}_{r_a,r_s} \\ \hline \mathcal{C}_{r_a} & \mathcal{D}_{r_a,r_s} \end{array}\right) = \left(\begin{array}{cc|ccc} A & B_{r_a}C_{Kr_a} & B_w & B_{f_a} & 0 \\ B_{Kr_s}C_{r_s} & A_K & B_{Kr_s}D_{wr_s} & 0 & B_{Kf_s} \\ \hline C_z & D_{zr_a}C_{Kr_a} & 0 & 0 & 0 \end{array}\right) \quad (5)$$

The closed-loop system via output-feedback is depicted in Figure 1 . Note that the controlled output under the failure condition is changed to z_{r_a}, vanishing the effect of actuator failure.

Now, it can be reformulated in the following framework. Define

$$\mathcal{K}_{r_a,r_s} := \begin{pmatrix} 0 & C_{Kr_a} \\ B_{Kr_s} & A_K \end{pmatrix} \quad (6)$$

Now in this stage, the Bounded Real Lemma must be satisfied for the reliable control problem in order to meet the $\mathcal{H}_\infty$ specification[3]. Since the expressions like $\mathcal{A}_r^T\mathcal{P} + \mathcal{P}\mathcal{A}_r$, however, involve products of $\mathcal{P}$ and the controller variables, the resulting problem is nonlinear. However, the LMI framework of the output-feedback control requires the matrix variables to change to linearize, unlike the state-feedback case, of which all inequalities are affine easily in $X = \mathcal{P}^{-1}$ and $V = K\mathcal{P}^{-1}$.

Let us partition $\mathcal{P}$ and $\mathcal{P}^{-1}$ as

$$\mathcal{P} = \begin{pmatrix} Y & N \\ N^T & \star \end{pmatrix}, \quad \mathcal{P}^{-1} = \begin{pmatrix} X & M \\ M^T & \star \end{pmatrix} \quad (7)$$

where X and Y are $n \times n$ and symmetric, and $\star$ is any arbitrary matrix. From $\mathcal{P}\mathcal{P}^{-1} = I$, we can lead to

$$\mathcal{P}\Pi_1 = \Pi_2 \text{ with } \Pi_1 := \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}, \quad \Pi_2 := \begin{pmatrix} I & Y \\ 0 & N^T \end{pmatrix} \quad (8)$$

Let us define the change of controller variables as follows:

$$\begin{aligned}\hat{A}_{r_a,r_s} &:= NA_KM^T + NB_{Kr_s}C_{r_s}X + YB_{r_a}C_{Kr_a}M^T + YAX \\ \hat{B}_{r_s} &:= NB_{Kr_s}, \quad \hat{B}_{f_s} := NB_{Kf_s}, \\ \hat{C}_{r_a} &:= C_{Kr_a}M^T, \quad \hat{C}_{f_a} := C_{Kf_a}M^T, \end{aligned} \quad (9)$$

The motivation for this transformation lies in the following identities:

$$\begin{aligned}\bar{\mathcal{A}}_{r_a,r_s} &:= \Pi_1^T\mathcal{P}\mathcal{A}_{r_a,r_s}\Pi_1 = \Pi_2^T\mathcal{A}_{r_a,r_s}\Pi_1 \\ &= \begin{pmatrix} AX + B_{r_a}\hat{C}_{r_a} & A \\ \hat{A}_{r_a,r_s} & YA + \hat{B}_{r_s}C_{r_s} \end{pmatrix} \\ \bar{\mathcal{B}}_{r_a,r_s} &:= \Pi_1^T\mathcal{P}\mathcal{B}_{r_a,r_s} = \Pi_2^T\mathcal{B}_{r_a,r_s} \\ &= \begin{pmatrix} B_w & B_{f_a} & 0 \\ YB_w + \hat{B}_{r_s}D_{wr_s} & YB_{f_a} & \hat{B}_{f_s} \end{pmatrix} \\ \bar{\mathcal{C}}_{r_a} &:= \mathcal{C}_{r_a}\Pi_1 = \begin{pmatrix} C_zX + D_{zr_a}\hat{C}_{r_a} & C_z \end{pmatrix} \\ \Pi_1^T\mathcal{P}\Pi_1 &= \begin{pmatrix} X & I \\ I & Y \end{pmatrix}\end{aligned}$$

By virtue of the Bounded Real Lemma, the requirement to stabilize the system with $\mathcal{H}_\infty$ constraint $\|T_{r_a,r_s}\|_\infty < \gamma$ is to satisfy two inequalities for above closed-loop matrices $\mathcal{A}_{r_a,r_s}$, $\mathcal{B}_{r_a,r_s}$, $\mathcal{C}_{r_a}$, and $\mathcal{D}_{r_a,r_s}$. By a congruence transformation with $diag(\Pi_1, I, I)$ on the lemma, we obtain the following synthesis in the LMIs.

$$\begin{pmatrix} \bar{\mathcal{A}}_{r_a,r_s} + \bar{\mathcal{A}}_{r_a,r_s}^T & \bar{\mathcal{B}}_{r_a,r_s} & \bar{\mathcal{C}}_{r_a}^T \\ \bar{\mathcal{B}}_{r_a,r_s}^T & -\gamma^2 I & 0 \\ \bar{\mathcal{C}}_{r_a} & 0 & -I \end{pmatrix}$$

$$
= \left(\begin{array}{cc|c|c}
\begin{array}{c} AX+XA^T \\ +B_{r_a}\hat{C}_{r_a}+(B_{r_a}\hat{C}_{r_a})^T \end{array} & \hat{A}^T_{r_a,r_s}+A & \star\star\star & \star \\
\hat{A}_{r_a,r_s}+A^T & \begin{array}{c} A^TY+YA \\ +\hat{B}_{r_s}C_{r_s}+(\hat{B}_{r_s}C_{r_s})^T \end{array} & \star\star\star & \star \\
\hline
B_w^T & (YB_w+\hat{B}_{r_s}D_{wr_s})^T & & 0 \\
B_{f_a}^T & (YB_{f_a})^T & -\gamma^2 I & 0 \\
0 & \hat{B}_{f_s}^T & & 0 \\
\hline
C_zX+D_{zr_a}\hat{C}_{r_a} & C_z & 0\ 0\ 0 & -I
\end{array}\right)
$$

$$
< \ 0 \qquad (10)
$$

$$
\Pi_1^T \mathcal{P} \Pi_1 = \begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0 \qquad (11)
$$

The controller is of at least order n, and if M and N have full row rank and $\hat{A}_{r_a,r_s}$, $\hat{B}_{r_a,r_s}$, $\hat{C}_{r_a}$, X, and Y are given, we can always compute controller matrices A_K, B_K, and C_K for an actuator/sensor failure.

Let us finally look at the problem in view of constructing *single* robust and reliable controller that stabilizes against *all* cases of admissible actuator/sensor failure. Suppose that the cardinal number of Ω_a is l_a and the cardinal number of Ω_s is l_s. Then, all possible number of admissible failure cases including no failure is $L = L_a \times L_s = 2^{l_a} \times 2^{l_s}$ and L Lyapunov matrices $\mathcal{P}_1, \cdots, \mathcal{P}_L$ corresponding to all failure cases are required to meet L LMIs (10) whose each one is assigned to the individual specification with (11). Since the expressions like $\mathcal{A}_{i,j}^T\mathcal{P}_{i,j} + \mathcal{P}_{i,j}\mathcal{A}_{i,j}$ for $i = 1, \cdots, L_a, j = 1, \cdots, L_s$, involve products of $\mathcal{P}_j$ and the controller variables, the resulting problem is nonconvex. Thus, two specific requirements are needed to recover the convexity. The first one is to have a single Lyapunov matrix $\mathcal{P}$ that satisfies $\mathcal{P}_1 = \cdots = \mathcal{P}_L = \mathcal{P}$. Second requirement is that, in order for the controller to conform to all specifications, the controller variables A_K, B_K, and C_K have to be same for all failure cases. Nonetheless, products of X, Y, and the controller variables could not be still eliminated in the matrix inequalities, nor assigned to new variables like V_j in the state-feedback case. Therefore, we propose the following theorem for the output-feedback problem in accordance with the state-feedback problem.

Theorem 1 Consider the linear system (2) with unreliable actuators u_{Ω_a} and unreliable sensors y_{Ω_s}, and assume $(A, B_{\bar{\Omega}_a}, C_{\bar{\Omega}_s})$ is a stabilizable and detectable pair. The system is robustly stabilizable against any susceptible actuator and sensor failure and the $\mathcal{H}_\infty$ constraint $\|T_{r_a,r_s}\|_\infty < \gamma$ is satisfied if there exist symmetric positive definite matrices X and Y such that the following LMIs are satisfied

$$
\begin{pmatrix}
\bar{\mathcal{A}}_{\bar{\Omega}_a,\bar{\Omega}_s}+\bar{\mathcal{A}}^T_{\bar{\Omega}_a,\bar{\Omega}_s} & \bar{\mathcal{B}}_{\bar{\Omega}_a,\bar{\Omega}_s} & \bar{\mathcal{C}}^T_{\bar{\Omega}_a} & \Lambda^T_{\Omega_a}+\Gamma^T_{\Omega_a}R_{\Omega_a} & \Xi^T_{\Omega_s}+\Upsilon^T_{\Omega_s}R_{w\Omega_s} \\
\bar{\mathcal{B}}^T_{\bar{\Omega}_a,\bar{\Omega}_s} & -\gamma^2 I & 0 & 0 & 0 \\
\bar{\mathcal{C}}_{\bar{\Omega}_a} & 0 & -I & 0 & 0 \\
\Lambda_{\Omega_a}+R_{\Omega_a}\Gamma_{\Omega_a} & 0 & 0 & -R_{\Omega_a} & 0 \\
\Xi_{\Omega_s}+R_{w\Omega_s}\Upsilon_{\Omega_s} & 0 & 0 & 0 & -R_{w\Omega_s}
\end{pmatrix}
< 0 \quad (12)
$$

$$
\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0
$$

where

$$
\begin{aligned}
R_{\Omega_a} &= R^T_{\Omega_a} = D^T_{z\Omega_a}D_{z\Omega_a} > 0, \\
R_{w\Omega_s} &= R^T_{w\Omega_s} = \frac{1}{\gamma^2}D_{w\Omega_s}D^T_{w\Omega_s} > 0, \\
\Lambda_{\Omega_a} &= \begin{pmatrix} B^T_{\Omega_a} & B^T_{\Omega_a}Y \end{pmatrix}, \ \Gamma_{\Omega_a} = \begin{pmatrix} \hat{C}_{\Omega_a} & 0 \end{pmatrix}, \\
\Xi_{\Omega_s} &= \begin{pmatrix} C_{\Omega_s}X & C_{\Omega_s} \end{pmatrix}, \ \Upsilon_{\Omega_s} = \begin{pmatrix} 0 & \hat{B}^T_{\Omega_s} \end{pmatrix}
\end{aligned}
$$

Proof: Consider the system (2) has an actual failures of f_a and f_s. To meet the robust specification against the failures, Eq. (10) has to be satisfied. Thus, the proof suffices to show that Eq. (12) implies Eq. (10).

By simple calculation, Eq. (10) is equivalent to

$$
\bar{\mathcal{A}}_{r_a,r_s} + \bar{\mathcal{A}}^T_{r_a,r_s} + \frac{1}{\gamma^2}\bar{\mathcal{B}}_{r_a,r_s}\bar{\mathcal{B}}^T_{r_a,r_s} + \bar{\mathcal{C}}^T_{r_a}\bar{\mathcal{C}}_{r_a} < 0 \quad (13)
$$

Now let's express (13) in terms of the maximum failure case, that is, $r_a = \bar{\Omega}_a$ and $r_s = \bar{\Omega}_s$. There must be pointed out that just one controller is needed for any susceptible failures. Since the controller matrices A_K, B_K, and C_K are all the same for such failures, $\hat{A}_{r_a,r_s}$, $\hat{B}_{r_s}$, $\hat{C}_{r_a}$, $\hat{A}_{\bar{\Omega}_a,\bar{\Omega}_s}$, $\hat{B}_{\bar{\Omega}_s}$, and $\hat{C}_{\bar{\Omega}_a}$ from (9) are closely related with the following equations.

$$
\begin{aligned}
\hat{A}_{r_a,r_s} &= \hat{A}_{\bar{\Omega}_a,\bar{\Omega}_s}+NB_{K\Omega_a-f_a}C_{\Omega_a-f_a}X+YB_{\Omega_a-f_a}C_{K\Omega_a-f_a}M^T \\
\hat{B}_{r_s} &= \begin{pmatrix} \hat{B}_{\bar{\Omega}_s} & NB_{K\Omega_s-f_s} \end{pmatrix} \\
\hat{C}_{r_a} &= \begin{pmatrix} \hat{C}_{\bar{\Omega}_a} \\ C_{K\Omega_a-f_a}M^T \end{pmatrix}
\end{aligned}
$$

Thus, (13) is rearranged to

$$
\begin{aligned}
&\bar{\mathcal{A}}_{\bar{\Omega}_a,\bar{\Omega}_s} + \bar{\mathcal{A}}^T_{\bar{\Omega}_a,\bar{\Omega}_s} + \frac{1}{\gamma^2}\bar{\mathcal{B}}_{\bar{\Omega}_a,\bar{\Omega}_s}\bar{\mathcal{B}}^T_{\bar{\Omega}_a,\bar{\Omega}_s} + \bar{\mathcal{C}}^T_{\bar{\Omega}_a}\bar{\mathcal{C}}_{\bar{\Omega}_a} \\
&+ \Lambda^T_{\Omega_a-f_a}\Gamma_{\Omega_a-f_a} + \Gamma^T_{\Omega_a-f_a}\Lambda_{\Omega_a-f_a} + \Gamma^T_{\Omega_a-f_a}R_{\Omega_a-f_a}\Gamma_{\Omega_a-f_a} \\
&+ \Xi^T_{\Omega_s-f_s}\Upsilon_{\Omega_s-f_s} + \Upsilon^T_{\Omega_s-f_s}\Xi_{\Omega_s-f_s} + \Upsilon^T_{\Omega_s-f_s}R_{w\Omega_s-f_s}\Upsilon_{\Omega_s-f_s} \\
&- \frac{1}{\gamma^2}\Lambda^T_{\Omega_a-f_a}\Lambda_{\Omega_a-f_a} - \frac{1}{\gamma^2}\Upsilon^T_{\Omega_s-f_s}R_{w\Omega_s-f_s}\Upsilon_{\Omega_s-f_s} < 0 \quad (14)
\end{aligned}
$$

where

$$
\begin{aligned}
R_{\Omega_a-f_a} &= R^T_{\Omega_a-f_a} = D^T_{z\Omega_a-f_a}D_{z\Omega_a-f_a} > 0, \\
R_{w\Omega_s-f_s} &= R^T_{w\Omega_s-f_s} = \frac{1}{\gamma^2}D_{w\Omega_s-f_s}D^T_{w\Omega_s-f_s} > 0, \\
\Lambda_{\Omega_a-f_a} &= \begin{pmatrix} B^T_{\Omega_a-f_a} & B^T_{\Omega_a-f_a}Y \end{pmatrix}, \ \Gamma_{\Omega_a-f_a} = \begin{pmatrix} \hat{C}_{\Omega_a-f_a} & 0 \end{pmatrix}, \\
\Xi_{\Omega_s-f_s} &= \begin{pmatrix} C_{\Omega_s-f_s}X & C_{\Omega_s-f_s} \end{pmatrix}, \ \Upsilon_{\Omega_s-f_s} = \begin{pmatrix} 0 & \hat{B}^T_{\Omega_s-f_s} \end{pmatrix}.
\end{aligned}
$$

Now in this stage, notice that the following relations are satisfied.

$$
(\Lambda_{\Omega_a-f_a}+R_{\Omega_a-f_a}\Gamma_{\Omega_a-f_a})^T R^{-1}_{\Omega_a-f_a}(\Lambda_{\Omega_a-f_a}+R_{\Omega_a-f_a}\Gamma_{\Omega_a-f_a})
$$

$$=(\Lambda_{\Omega_a}+R_{\Omega_a}\Gamma_{\Omega_a})^T R_{\Omega_a}^{-1}(\Lambda_{\Omega_a}+R_{\Omega_a}\Gamma_{\Omega_a})$$
$$-(\Lambda_{f_a}+R_{f_a}\Gamma_{f_a})^T R_{f_a}^{-1}(\Lambda_{f_a}+R_{f_a}\Gamma_{f_a}) \quad (15)$$
$$(\Xi_{\Omega_s-f_s}+R_{w\Omega_s-f_s}\Upsilon_{\Omega_s-f_s})^T R_{w\Omega_s-f_s}^{-1}(\Xi_{\Omega_s-f_s}+R_{w\Omega_s-f_s}\Upsilon_{\Omega_s-f_s})$$
$$=(\Xi_{\Omega_s}+R_{w\Omega_s}\Upsilon_{\Omega_s})^T R_{w\Omega_s}^{-1}(\Xi_{\Omega_s}+R_{w\Omega_s}\Upsilon_{\Omega_s})$$
$$-(\Xi_{f_s}+R_{wf_s}\Upsilon_{f_s})^T R_{wf_s}^{-1}(\Xi_{f_s}+R_{wf_s}\Upsilon_{f_s}) \quad (16)$$

After completing the square and using the above relation, the above equation is changed to the following equation.

$$\bar{\mathcal{A}}_{\bar{\Omega}_a,\bar{\Omega}_s} + \bar{\mathcal{A}}^T_{\bar{\Omega}_a,\bar{\Omega}_s} + \frac{1}{\gamma^2}\bar{\mathcal{B}}_{\bar{\Omega}_a,\bar{\Omega}_s}\bar{\mathcal{B}}^T_{\bar{\Omega}_a,\bar{\Omega}_s} + \bar{\mathcal{C}}^T_{\bar{\Omega}_a}\bar{\mathcal{C}}_{\bar{\Omega}_a}$$
$$+(\Lambda_{\Omega_a}+R_{\Omega_a}\Gamma_{\Omega_a})^T R_{\Omega_a}^{-1}(\Lambda_{\Omega_a}+R_{\Omega_a}\Gamma_{\Omega_a})$$
$$+(\Xi_{\Omega_s}+R_{w\Omega_s}\Upsilon_{\Omega_s})^T R_{w\Omega_s}^{-1}(\Xi_{\Omega_s}+R_{w\Omega_s}\Upsilon_{\Omega_s})$$
$$-(\Lambda_{f_a}+R_{f_a}\Gamma_{f_a})^T R_{f_a}^{-1}(\Lambda_{f_a}+R_{f_a}\Gamma_{f_a})$$
$$-\Lambda^T_{\Omega_a-f_a}(R^{-1}_{\Omega_a-f_a}+\frac{1}{\gamma^2}I)\Lambda_{\Omega_a-f_a}$$
$$-(\Xi_{f_s}+R_{wf_s}\Upsilon_{f_s})^T R_{wf_s}^{-1}(\Xi_{f_s}+R_{wf_s}\Upsilon_{f_s})$$
$$-\Xi^T_{\Omega_s-f_s}R^{-1}_{w\Omega_s-f_s}\Xi_{\Omega_s-f_s}-\frac{1}{\gamma^2}\Upsilon^T_{\Omega_s-f_s}\Upsilon_{\Omega_s-f_s} < 0 \quad (17)$$

Since the given LMI condition (12) means that the six terms from the left hand side of Eq. (17) are less than zero, the inequality is always satisfied. Thus the proof is completed. □

After solving the above LMIs, we can find arbitrary nonsingular matrices M, N to satisfy $MN^T = I - XY$. Then M and N have full row rank when $I - XY$ is invertible. Invertibility of $I - XY$ is satisfied by Eq. (12). Finally we can always compute the controller by

$$C_K = \begin{pmatrix} C_{K\bar{\Omega}_a} \\ C_{K\Omega_a} \end{pmatrix} = \begin{pmatrix} \hat{C}_{\bar{\Omega}_a} \\ \hat{C}_{\Omega_a} \end{pmatrix} M^{-T} \quad (18)$$
$$B_K = \begin{pmatrix} B_{K\bar{\Omega}_s} & B_{K\Omega_s} \end{pmatrix} = N^{-1}\begin{pmatrix} \hat{B}_{\bar{\Omega}_s} & \hat{B}_{\Omega_s} \end{pmatrix}$$
$$A_K = N^{-1}(\hat{A}_{\bar{\Omega}_a,\bar{\Omega}_s} - NB_{K\bar{\Omega}_s}C_{\bar{\Omega}_s}X - YB_{\bar{\Omega}_a}C_{K\bar{\Omega}_a}M^T - YAX)M^{-T}$$

Example 1 Consider the MIMO linear uncertain system with two inputs and three outputs. The following system is an extension of one used in Veillette *et al.* (1992)[5] and Seo (1996)[4].

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} w(t) \quad (19)$$
$$z(t) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0.008 & 0 & -0.008 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w(t)$$

The spectrum of A is given by $spec(A) = \{-1.3160 \pm 2.9194i, 0.1906, -2.5585\}$. Note that the nominal system has an unstable mode.

Now, we try to design reliable output-feedback controller for both actuator and sensor failures. The corresponding LMI formulation is subjected to the LMIs (12) listed in Theorem 1. After minimizing $trace(X) + trace(Y)$ with $\gamma = 30$, we can obtain such a controller for which all LMI-related computations were performed from the *LMI Control Toolbox* (Gahinet *et al.* 1995)[1].

$$\dot{\zeta}(t) = 10^6 \begin{bmatrix} -1.045 & 1.227 & -1.120 & 0.448 \\ -1.606 & 1.885 & -1.720 & 0.689 \\ -1.058 & 1.242 & -1.134 & 0.454 \\ -0.674 & 0.791 & -0.722 & 0.289 \end{bmatrix} \zeta(t) + \begin{bmatrix} 4.938 & 5389 & 5151 \\ 7.577 & 8278 & 7914 \\ 4.989 & 5455 & 52158 \\ 3.185 & 3476 & 3323 \end{bmatrix} y(t)$$
$$u(t) = \begin{bmatrix} 0.586 & -10.619 & 0.511 & -0.240 \\ -10.131 & 0.141 & -0.107 & 0.038 \\ -1.188 & 1.141 & -0.920 & -9.417 \end{bmatrix} \zeta(t) \quad (20)$$

III. Concluding Remarks

We solved the reliable $\mathcal{H}_\infty$ control problem for output-feedback case in the context of the LMI approach. The proposed control methodology does not only guarantee the robust stability, but also the target system is reliable in spite of the allowable actuator and sensor failures.

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