Stochastic Comparisons of Markovian Retrial Queues

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Abstract

We consider a Markovian retrial queueing model with service facility. Cusomers arrive from outside according to a Poisson process with rate λ. An arriving customer who finds k customers in the service facility enters the facility with probability p_k or joins the orbit with probability $1-p_k$ and retries to get service after exponential time. If returning customer from orbit finds k customers in the service facility, then it enters the facility with probability u_k or joins the orbit again with probability $v_k=1-u_k$ and retries its service after exponential time. Let θ_k be the total retrial rate when k customers are in the orbit. When there are n customers at the service facility, we assume that, in the absence of arrivals, the time until the next service completion is exponential with rate μ_n.

We are interested in the bivariate process $X(t)=(C(t),N(t))$ called the *CN-process*, where $C(t)$ and $N(t)$ represent the numbers of customers in the service facility and the orbit, respectively, at time t. Obviously, the bivariate process $\{X(t),\ t\geq 0\}$ is a Markov chain with the lattice set $E=Z^+\times Z^+$ where $Z^+=\{0,1,2,\cdots\}$, as the state space.

In this paper, instead of studying a performance measure in a quantitative fashion, we attempt to investigate a relationship between the sample paths of two *CN-processes* from the relationship between parameters by constructing equivalent processes on a common probability space.

Define a relation $<$ on E by $(i,j)<(k,l)$ if and only if $j\leq l$ and $i+j\leq k+l$, then it is immediate that $<$ is a partial order on E. If two stochastic processes Y and X are defined on the same probability space then X is almost surely smaller than Y with respect to $<$, written $X<_{as}Y$

if $P(X(t)<Y(t)$ for *all* $t\geq 0)=1$. Note that if $X<_{as}Y$ and Y and X have weak limits $\hat{X}$ and $\hat{Y}$, respectively, then $\hat{X}<\hat{Y}$ in distribution.

Theorem 1. Let $X^{(i)}=\{X^{(i)}(t),\ t\geq 0\}$, $i=1,2$ be the *CN* processes described in the previous section with arrival rates $\lambda^{(i)}$, service rates $\mu_n^{(i)}$ with $\mu_0^{(i)}=0$, retrial rates $\theta_n^{(i)}$ with $\theta_0^{(i)}=0$

and the probabilities $p_n^{(i)}$ and $u_n^{(i)}$ of entering the service facility of customers from outside and from the orbit, respectively. Suppose that $X^{(i)}$, $i=1,2$ are regular and

$$\begin{aligned}
&(i)\ \ X^{(1)}(0)=(i,j)<X^{(2)}(0)=(i',j')\\
&(ii)\ \ \lambda^{(1)}\le\lambda^{(2)}\\
&(iii)\ \ \mu_n^{(1)}\ge\mu_n^{(2)} \text{ and } \mu_n^{(i)}\le\mu_{n+1}^{(i)},\ i=1,2,\ n=0,1,2,\cdots,\\
&(iv)\ \ \theta_n^{(1)}\ge\theta_n^{(2)}\ n\ge 0,\\
&(v)\ \ p_n^{(1)}\ge p_n^{(2)} \text{ and } p_n^{(i)}\ge p_{n+1}^{(i)},\ i=1,2,\ n=0,1,2,\cdots,\\
&(vi)\ \ u_n^{(1)}\ge u_n^{(2)} \text{ and } u_n^{(i)}\ge u_{n+1}^{(i)},\ i=1,2,\ n=0,1,2,\cdots.
\end{aligned}$$

Then there are *CN* processes $\tilde{X}^{(i)}$, $i=1,2$ which are equivalent to $X^{(i)}$, $i=1,2$, respectively, on a common probability space such that

$$\tilde{X}^{(1)}<_{as}\tilde{X}^{(2)}.$$ □

Corollary 1. Let $\Sigma(c,K,M)$ be the retrial queue with parameters $\mu_n=\min(c,n)\mu$, $\theta_n=\min(M,n)\theta$ and $p_n=u_n=1$ for $n\le K-1$ and $p_n=u_n=0$ for $n\ge K$. Let $\lambda^{(k)}$, $\mu_n^{(k)}$ and $\theta_n^{(k)}$ be the corresponding parameters of $\Sigma(c_k,K_k,M_k)$, $k=1,2$ satisfying $\lambda^{(1)}\le\lambda^{(2)}$, $\mu^{(1)}\ge\mu^{(2)}$ and $\theta^{(1)}\ge\theta^{(2)}$. Then for $c_1\ge c_2$, $K_1\ge K_2$ and $M_1\ge M_2$ and the initial values (i_k,j_k) of the systems $\Sigma(c_k,K_k,M_k)$, $k=1,2$, satisfying $(i_1,j_1)<(i_2,j_2)$, we have

$$\Sigma(c_1,K_1,M_1)<_{as}\Sigma(c_2,K_2,M_2)$$

where $<_{as}$ means that the corresponding *CN*-processes are related by the partial order $<_{as}$. □

Corollary 2. Let $\tilde{\Sigma}(c_k,K_k,M_k)$ (with $K\le M$) be the retrial queue with c_k parallel servers and waiting space K_k including service space in which the intensity of retrial becomes infinity as soon as the number of customers in orbit reaches some level M_k and with arrival rate $\lambda^{(k)}$, service rate $\mu^{(k)}$ of each server and retrial rate $\theta^{(k)}$, $k=1,2$ satisfying $\lambda^{(1)}\le\lambda^{(2)}$, $\mu^{(1)}\ge\mu^{(2)}$ and $\theta^{(1)}\ge\theta^{(2)}$.
Let (i_k,j_k) be the initial states of the system $\tilde{\Sigma}(c_k,K_k,M_k)$, $k=1,2$. If $c_1\ge c_2$, $K_1\ge K_2$, $M_1\le M_2$, and $(i_1,j_1)<(i_2,j_2)$, then we have

$$\tilde{\Sigma}(c_1,K_1,M_1)<_{as}\tilde{\Sigma}(c_2,K_2,M_2).$$ □

Corollary 3. (Comparisons of impatient customers) Let $\Sigma_I^{(i)}(c,K,M_i)$ be retrial queues with impatient customers and with parameters $\lambda^{(i)}$, $\mu^{(i)}$, $\theta^{(i)}$, $\alpha^{(i)}$ and $\beta^{(i)}$, $i=1,2$. Suppose that $\lambda^{(1)}\le\lambda^{(2)}, \mu^{(1)}\ge\mu^{(2)}, \theta^{(1)}\ge\theta^{(2)}, \alpha^{(1)}\le\alpha^{(2)}$ and $\beta^{(1)}\le\beta^{(2)}$. Then for $M_1\ge M_2$, we have

$$\Sigma_I^{(1)}(c,K,M_1)<_{as}\Sigma_I^{(2)}(c,K,M_2).$$ □

Corollary 4. (Convergence of stationary distributions) We assume that the stability condition $\rho=\frac{\lambda}{c\mu}<1$ and let $\widetilde{p_{ij}}^{(M)}$, p_{ij} and $p_{ij}^{(M')}$ be the stationary distributions of the *CN*-processes in $\tilde{\Sigma}(c,K,M)$, $\Sigma(c,K,\infty)$ and $\Sigma(c,K,M')$, respectively. Then we have the relation

$$\{\widetilde{p_{ij}}^{(M)}\}<\{p_{ij}\}<\{p_{ij}^{(M')}\},\ M,M'\ge 0$$

and

$$\lim_{M\to\infty}\widetilde{p_{ij}}^{(M)}=p_{ij}=\lim_{M\to\infty}p_{ij}^{(M)},\ 0\le i\le K,\ j\ge 0.$$ □