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# 리프팅을 이용한 비선형 웨이블릿 변환

이창수, 유경렬
한양대학교 전기공학과

# Nonlinear Wavelet Transform Using Lifting

Chang Soo Lee, Kyung Yul Yoo
Dept. of Electrical Eng., Hanyang University

**Abstract** - This paper introduces a nonlinear wavelet transform based on the lifting scheme, which is applied to signal denoising through the translation invariant wavelet transform. The wavelet representation using orthogonal wavelet bases has received widespread attention. Recently the lifting scheme has been developed for the construction of biorthogonal wavelets in the spatial domain. In this paper, we adaptively reduce the vanishing moments in the discontinuities to suppress the ringing artifacts and this customizes wavelet transforms providing an efficient framework for the translation invariant denoising. Special care has been given to the boundaries, where we design a set of different prediction coefficients to reduce the prediction error.

## 1. Introduction

The wavelet transform is an atomic decomposition that represents a real-valued continuous-time signal $x(t)$ in terms of shifted and dilated versions of a prototype bandpass wavelet function $\psi(t)$ and lowpass scaling function $\phi(t)$ [1]. For special choices of the wavelet and scaling function, the atoms

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t-k), \quad j,k\in Z,\ j\le J \tag{1}$$

$$\phi_{J,k}(t) = 2^{J/2}\phi(2^J t-k) \tag{2}$$

form an orthonormal basis, and $x(t)$ could be decomposed into

$$x(t) = \sum_k c_k\phi_{J,k}(t) + \sum_{j=0}^{J}\sum_k d_{j,k}\psi_{j,k}(t) \tag{3}$$

with $d_{j,k}=\int x(t)\psi_{j,k}(t)dt$ and $c_k=\int x(t)\phi_{J,k}(t)dt$. The wavelet coefficients $\{d_{j,k}\}$ and scaling coefficients $\{c_k\}$ comprise the wavelet transform. The set of scaling coefficients $\{c_k\}$ represents coarse signal information at scale $j=0$, whereas the set of wavelet coefficients $\{d_{j,k}\}$ represents detail information at scales $j=1,2,\cdot\cdot\cdot,J$.

The representation of the data using wavelets coefficients offers an accurate approximation of $f$ by using only a few wavelet coefficients. It comes from the fact that the vanishing moments property of wavelets suppresses low-order polynomial signals in the highpass filter and we get a small fraction of wavelet coefficients [2, 3].

The lifting scheme is a flexible tool for constructing wavelets without employing the Fourier transform and could therefore build wavelet bases over non-translation invariant domains such as bounded regions of $R^P$ or surfaces. The lifting algorithm is asymptotically twice as fast as the standard DWT algorithm and allows a fully in-place calculation of the wavelet transform without allocating auxiliary memory. The inverse wavelet transform could be found simply by undoing the operations of the forward transform. Also all wavelet transforms could be factored into the lifting steps with multiple predicts and updates [3, 4].

This paper concerns the application of lifting scheme to signal denoising. We introduce a nonlinear lifting which reduces the vanishing moments in the discontinuities of the data to customize the DWT. Through the nonlinear lifting, we can suppress the ringing artifacts near the discontinuities using the translation invariant denoising [5]. This paper is organized as follows. Section 2 provides the basics on the lifting scheme and discusses the design procedure for the predict and update stage. In Section 3, we review the translation invariant wavelet denoising algorithm and apply the nonlinear lifting to signal denoising. Finally Section 4 contains concluding remarks and future works.

## 2. Lifting scheme

The lifting scheme could be used in situations where the Fourier transform is difficult to apply. The lifting is a new method for constructing wavelets. Three steps of the lifting are described on Fig. 1. $\{\gamma_{-j}\}$ are computed by successively applying these three stages and represent the wavelet coefficients. $\{\lambda_{-j}\}$ are also lifted based on these wavelet coefficients and denote the scaling coefficients.

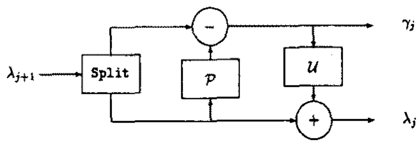


Fig. 1. Structure of lifting scheme : Split, Predict, and Update

### 2.1 Split

In this stage, we divide the signal into the even set $\{\lambda_{-j}\}$ and the odd set $\{\gamma_{-j}\}$. This mechanism is shown in Fig. 2. At each level $j$, $\lambda_{-(j-1),2k}$ and $\lambda_{-(j-1),2k+1}$ are set to $\lambda_{-j,k}$ and $\gamma_{-j,k}$ respectively. In the simulation, we don't split the data physically, but the split and predict stages are combined into one function because of the in-place calculation [2, 3].

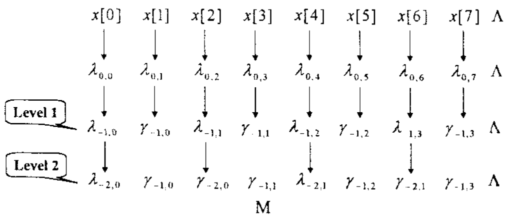


Fig. 2. Schematic diagram explaining the split stage

### 2.2 Predict

The prediction operator $P$ is based on the polynomial interpolation of order $N-1$ to find the predicted values. Through this prediction, we could suppress signals which resemble polynomials of order up to $N-1$ [6]. For the cubic interpolation, we have four cases:

- Near the left boundary

Case I. $1\lambda$ on the left and $3\lambda$'s on the right

- Middle

Case II. $2\lambda$'s on the left and $2\lambda$'s on the right

- Near the right boundary

Case III. $3\lambda$'s on the left and $1\lambda$ on the right

Case IV. $4\lambda$'s on the left and $0\lambda$'s on the right

To find the prediction coefficients, we first design a prediction operator $P$ so as to have better approximation of the signal of order up to $N-1$ [7]. For $N=4$, note that all polynomials of order up to $N-1$ would be suppressed if

$$\begin{bmatrix} -3^0 & -2^0 & -1^0 & 0^0 & 1^0 & 2^0 & 3^0 \\ -3^1 & -2^1 & -1^1 & 0^1 & 1^1 & 2^1 & 3^1 \\ -3^2 & -2^2 & -1^2 & 0^2 & 1^2 & 2^2 & 3^2 \\ -3^3 & -2^3 & -1^3 & 0^3 & 1^3 & 2^3 & 3^3 \end{bmatrix} \begin{bmatrix} -p_1 \\ 0 \\ -p_2 \\ 1 \\ -p_3 \\ 0 \\ -p_4 \end{bmatrix} = \mathbf{O}_{4\times1} \quad (4)$$

Eq. (7) is simplified into

$$Vp = b, \quad (5)$$

where $V=\begin{bmatrix} -3^0 & -1^0 & 1^0 & 3^0 \\ -3^1 & -1^1 & 1^1 & 3^1 \\ -3^2 & -1^2 & 1^2 & 3^2 \\ -3^3 & -1^3 & 1^3 & 3^3 \end{bmatrix}$,

$$b=\begin{cases} [\ -2^0,\ -2^1,\ -2^2,\ -2^3\ ]^T, & \text{for case I} \\ [\ 1,\ 0,\ 0,\ 0\ ]^T, & \text{for case II} \\ [\ 2^0,\ 2^1,\ 2^2,\ 2^3\ ]^T, & \text{for case III} \\ [\ 4^0,\ 4^1,\ 4^2,\ 4^3\ ]^T, & \text{for case IV} \end{cases}, \text{ and}$$

$p\in(4\times1)$ is the 4-point prediction coefficient matrix.

Near the discontinuities, we reduced the number of prediction points because the low vanishing moments should be used to reduce the ringing artifacts (pseudo-Gibbs oscillations) [1], and the lifting became nonlinear.

† The boundary cases are not considered.

### 2.3 Update

The update stage preserves all the polynomials of order $N-1$. To find the update coefficients, we first initialize the integral-moment matrix $m^j\in(N\times L)$, where $L$ is the length of the signal. For $N=4$ at level $j=0$,

$$m^j = [m_0^j\ m_1^j\ m_2^j\ m_3^j\ \cdots] = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & \cdots \\ 0 & 1 & 4 & 9 & \cdots \\ 0 & 1 & 8 & 27 & \cdots \end{bmatrix} \quad (6)$$

Once all the moments have been initialized, the moments corresponding to the $\lambda$'s have to be updated to preserve the average at every level. The idea is that each $\gamma$ coefficient would give back to the $\lambda$'s that were used to predict it as it received, and this amount is given by the prediction coefficients [6].

$$m_{2k+2\ell}^j := m_{2k+2\ell}^j + p_{2,\ell+1} m_{2k+1}^j, \quad \ell=0,1,2,3 \ ^{\dagger} \quad (7)$$

To find the update coefficients for every $\gamma$, we solve an equation:

$$[m_{2k-2}^j\ m_{2k}^j\ m_{2k+2}^j\ m_{2k+4}^j]\, u_k^j = {m_{2k+1}^j}^{T} \ ^{\dagger} \quad (8)$$

Finally, we update $\lambda$'s based on $\gamma$'s with the update coefficients. For each $\gamma$,

$$\lambda_{-j,k+l-1} := \lambda_{-j,k+l-1} + u_{l,k}^j\, \gamma_{-j,k}, \quad l=0,1,2,3 \ ^{\dagger} \quad (9)$$

## 3. Translation invariant denoising

As one easily knows, the discrete wavelet transform is not translation invariant: i.e., there is no simple relationship between the wavelet coefficients of the original and the shifted signal. In this section, we present a translation-invariant DWT using an undecimated filter bank [5]. The main advantage of the wavelet transform without subsampling is the translation invariance of the coefficients

With the traditional (orthogonal, maximally decimated) denoising, we suffer from visual artifacts: pseudo-Gibbs oscillations in the neighborhood of discontinuities are caused by the lack of translation invariance of the wavelet basis. When using Haar wavelets, a discontinuity precisely at location $n/2$ will lead to essentially no pseudo-Gibbs oscillations; a discontinuity near a binary irrational like $n/3$ will lead to significant pseudo-Gibbs oscillations. To suppress these artifacts, we've chosen the translation-invariant denoising [5].

Let $X=Wx$ be the (orthogonal) DWT of $x$ and $S_R$ be a matrix performing a circular right shift by $R$ with $R\in Z$. Then

$$X_s = Wx_s = WS_Rx = WS_R W^{-1} X \quad (22)$$

which establishes the connection between the wavelet transforms of two shifted versions of a signal, $x$ and $x_s$, by the orthogonal matrix $WS_R W^{-1}$. Using these transforms, all circular shifts of the input signal are calculated and the denoised output signals are averaged in the reconstruction [5].

Four signals, Blocks, Bumps, Heavy Sine, and Doppler [5] were tested for signal denoising. Each signal had 2048 samples and the standard deviation of them was 7. The signals were corrupted by a white Gaussian noise with the standard deviation 1. In Table 1, the root mean square error (RMSE) performance of the DWT using Daubechies-8 (db8) and the nonlinear lifting is compared. The RMSEs in Table 1 show that the nonlinear lifting is close to the Haar when the signal contains the discontinuities like a Blocks.

Table 1. RMSEs of signal denoising

| Signal | RMSE | | |
|---|---|---|---|
| | db8 | Haar | Nonlinear lifting |
| Blocks | 14.0623 | 6.8720 | 8.7683 |
| Bumps | 14.1334 | 16.1131 | 15.4897 |
| HeaviSine | 7.8586 | 8.3801 | 7.8649 |
| Doppler | 10.6978 | 16.0959 | 11.6372 |

In Fig. 3 and 4, four signals are translation invariant denoised through the nonlinear lifting.

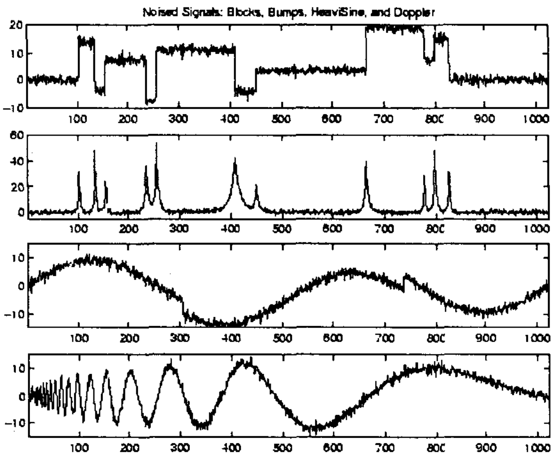


Fig 3. Noised signals

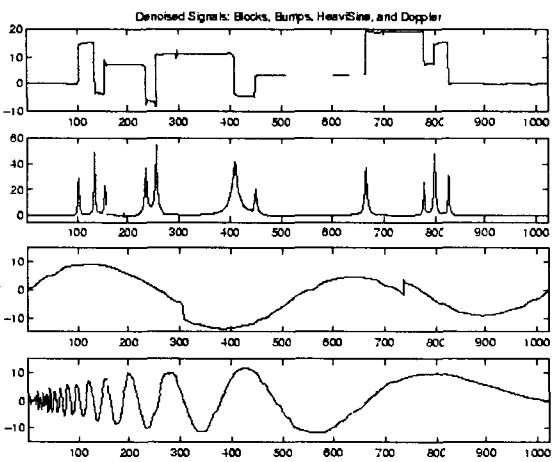


Fig. 4. Denoised signals

## 4. Conclusions

In this paper, we have introduced a nonlinear DWT based on the lifting. Using the lifting scheme, it is particularly easy to adapt the DWT to the signals. The nonlinear lifting is shown to represent both smooth and edgy signal elements.

For good performance, the shorter wavelets are required in the neighborhood of singularities, whereas the longer wavelets with more vanishing moments could improve the approximation in regions where the signal is more regular [8].

In terms of results, the adaptive lifting becomes the Haar-like DWT when the signal has singularities. In contrast, its performance is close to Daubechies 8 when the signal is more regular.

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