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퍼지넘버의 엔트로피 연산에 관한 연구

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ENTROPY ARITHMETIC OPERATIONS OF FUZZY NUMBERS

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Abstract

There have been several tipical methods being used to measure the fuzziness (entropy) of fuzzy sets. Pedrycz is the original motivation of this paper. This paper studies the entropy variation on the fuzzy numbers with arithmetic operations(addition, subtraction, multiplication). It is shown that through the arithmetic operations, the entropy of the resultant fuzzy number has the arithmetic relation with the entropy of each original fuzzy number. This paper generalize earlier results of Pedrycz [FSS 64(1994) 21-30] and Wang and Chiu [FSS 103(1999) 443-455].

1. Introduction

The entropy of a fuzzy set is a measure of fuzziness of the fuzzy set. Since Shannon and Weaver[10] used of entropy in information theory in 1964, there have been several typical methods used to measure the fuzziness of fuzzy sets is possibility theory (see De Luca and Termini[2], Kaufmann[6], Yager[15]). And there is lots of literature talking about the entropies of fuzzy sets (see [1, 3, 4, 8, 14]). One rather interesting application of the entropy measure for a fuzzy set was recently introduced by Pedrycz [FSS 64(1994) 21-30]. Pedrycz showed the entropy of triangular fuzzy number change when the interval size of the support is changed. Wang and Chiu[12] extended the result of Pedrycz[9] to the other types of fuzzy sets. Wang and Chiu[12] also considered the relationship between the entropies of the resultant three common types of fuzzy numbers and the original three common types of fuzzy numbers through arithmetic operations including addition, subtraction and multiplication. For the triangular fuzzy numbers, these results has been proposed by Wang and Chiu[11]. In this paper, we will have far-reaching generalizations of the results of Pedrycz[9] and Wang and Chiu[11. 12]. Indeed, we will show the relationship between the entropies of the resultant fuzzy number and the original fuzzy numbers without any restriction of types of fuzzy numbers.

2. Preliminary

Let X be real and finite.

Consider a fuzzy set and its associated membership function $A : X \to [0, 1]$. As studied in [2], the entropy of A, determined at a fixed element X is defined as

$$H(A(x)) = h(A(x)) \tag{2.1}$$

where $h : [0, 1] \to [0, 1]$ is monotonically increasing in [0, ½] and monotonically decreasing on the other half of the unit interval; moreover, $h(u)=0$, as $u=0$ and 1; and $h(u)=1$ as $u=\frac{1}{2}$. Typical examples of $h(u)$ include

$$h(u)=u\log_2 u, \tag{2.2}$$

$$h(u)=\begin{cases} 2u & \text{if } u\in[0,\frac{1}{2}] \\ 2(1-u) & \text{if } u\in[\frac{1}{2},1] \end{cases}, \tag{2.3}$$

$$h(u)=4u(1-u), \tag{2.4}$$

and

$$h(u)=-u\ln u-(1-u)\ln(1-u), \tag{2.5}$$

where (2.5) is called Shannon's function[10].

To determine a global entropy $H(A)$ of the fuzzy set A independent of x, the above expression has to be aggregated over the entire universe of discourse

X as follows[9]:

$$H(A)=\int_X h(A(x))p(x)dx,$$

where $p(x)$ is the probability density function of the available data in X. It is known that the larger $H(A)$ is, the more is the fuzziness of the fuzzy set A.

For the membership function of the fuzzy set $A(x)$, A^α denotes the α-cut of A , i.e. $A^\alpha \triangleq \{x|A(x)\geq\alpha,\ x\in X\}$; $A^{\alpha'}$ is the strong α-cut of A, i.e. $A^{\alpha'}\triangleq\{x|A(x)>\alpha,\ x\in X\}$. If $\alpha=0$, $A^{0'}$ is called the "support" of A.

Definition 1. Suppose we have any two fuzzy sets A_1 and A_2 with the support $A_1^{0'}=(a_1, B_1)\subset X$ and $A_2^{0'}=(a_2, b_2)\subset X$, respectively. If

$$A_1(\check{x})=A_2(\hat{x}), \tag{2.6}$$

where

$\check{x}=a_1+c_1, \hat{x}=a_2+(c_1/(b_1-a_1)), \check{x}\in[a_1, b_1]$ and $\hat{x}\in[a_2, b_2]$;

or

$$A_1(\check{x})=A_2(\hat{x}), \tag{2.7}$$

where $\check{x}=a_1+c_1$, $\hat{x}=b_2-(c_1/(b_1-a_1))(b_2-a_2)$, $\hat{x}\in[a_2, b_2]$. Then we call A_1 and A_2 to be the "same type of fuzzy sets".

It is also noted that if $b_2=-a_1$, $a_2=-b_1$, then let A_2 be called the "image" of A_1 and be denoted by A_1^-, where A_1 and A_2 also satisfy (2.7).

Let us recall the notations of arithmetic operations on fuzzy numbers. A fuzzy set A is called a "fuzzy number" if it is convex and normal. Suppose A, B are two fuzzy numbers; we have [7]

$$(A\otimes B)^\alpha \triangleq A^\alpha \otimes B^\alpha, \alpha\in(0,1]; \tag{2.8}$$

moreover,

$$_\alpha A(x)\triangleq\begin{cases}\alpha & x\in A^\alpha,\\ 0 & x\notin A^\alpha.\end{cases}$$

Then by the decomposition theorem[7], we have

$$A\otimes B\triangleq \bigcup_{\alpha\in[0,1]} {}_\alpha(A\otimes B), \tag{2.9}$$

where $\otimes$ denotes any arithmetic operation which may be addition, subtraction, multiplication, and division. $\bigcup$ denotes the standard fuzzy union. We consider here three basic arithmetic operations: addition (i.e. $A+B$), subtraction(i.e. $A-B$), and a simple multiplication (i.e. $k\cdot A$ where k is a constant).

There is a remarkably simple and useful representation of Lebesgue integrals over subsets of R in terms of Riemann-Stieltjes integrals. Let

$$w(\alpha)=w_{f,E}(\alpha)=|\{x\in E \mid f(x)>\alpha\}|,$$

where f is a measurable function on E, $-\infty<\alpha<+\infty$ and $|\cdot|$ is the Lebesgue measure of the set $\cdot$. We call $w_{f,E}$ the distribution function of f on E. Some properties of w were given in [13]. The following result is the essential tool of this paper.

Theorem 1[13]. *If $a<f\leq b$ (a and b finite) in E and Φ is continuous on $[a, b]$, then*

$$\int_E \Phi(f)=-\int_a^b \Phi(\alpha)\,dw(\alpha).$$

The next result is the very useful formula for integration by parts.

Theorem 2[13]. If $\int_a^b f\,d\Phi$ exists, then so does $\int_a^b \Phi\,df$, and $\int_a^b f\,d\Phi = [f(b)\Phi(b) - f(a)\Phi(a)] - \int_a^b \Phi\,df$.

We write $\{f>\alpha\} = \{x\in E \mid f(x) > \alpha\}$, ect.

3. Main results

Lemma 1. Let A_i, $i=1,2,\cdots,n$ be fuzzy numbers. Then

$$|\{A_1+A_2+\cdots+A_n>\alpha\}|=|\{A_1>\alpha\}|+\cdots+|\{A_n>\alpha\}|$$

for any $\alpha\in(0, 1]$.

Lemma 2. Let A_1 and A_2 be two fuzzy numbers. Then, for any $\alpha\in(0, 1]$

$$|\{A_1-A_2>\alpha\}|=|\{A_1>\alpha\}|+|\{A_2>\alpha\}|.$$

Lemma 3. Let A_1 and A_2 be two fuzzy numbers. Then, for any $\alpha \in (0, 1)$

$$|\{A_1 \cdot A_2 > \alpha\}| = |\{A_1 > \alpha\}||\{A_2 > \alpha\}|.$$

Lemma 4. For a constant k and a fuzzy number A, we have

$$|\{kA > \alpha\}| = k\, |\{A > \alpha\}|.$$

Theorem 3. Let X be a bounded set in R and A_i, $i=1,2,\cdots,n$, be fuzzy numbers such that the support of $A_i \subset X$, $i=1,2,\cdots,n$ and the support of $\sum_{i=1}^{n} A_i \subset X$. Suppose that $p(s)=s$, where s is a constant over X, then we have

$$H(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} H(A_i).$$

Theorem 4. Under the same conditions of Theorem 3, we have

$$H(\sum_{i\in I_1} A_i - \sum_{j\in I_2} A_j) = \sum_{k=1}^{n} H(A_k)$$

where I_1 and I_2 are two crisp sets of digits and $I_1 \cup I_2 = \{1,2,\cdots,n\}$, $I_1 \cap I_2 = \emptyset$.

Theorem 5. Let A be any fuzzy numbers with bounded support and k be any constant. Then

$$H(k \cdot A) = k \cdot H(A).$$

Theorem 6. Let X be a bounded set in R and A_i, $i=1,2,\cdots,n$ be fuzzy numbers such that the support of $A_i \subset X$, $i=1,2,\cdots,n$ and the support of $A_1 \cdot A_2 \cdots A_n \subset X$. Suppose that $p(s)=s$, where s is a constant over X and $|A_i > \alpha| = w_i(\alpha)$, $i=1,2,\cdots,n$, $h(\alpha) = |h > \alpha|$.

$$H(A_1 \cdot A_2 \cdots A_n) = s\int_0^1 A_1(\alpha)A_2(\alpha)\cdots A_n(\alpha)\,dh(\alpha)$$

$$(= s\int_0^1 A_1(\alpha)A_2(\alpha)\cdots A_n(\alpha)\,h'(\alpha)\,d\alpha$$

if h is continuously differentiable on [0, 1])

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