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Computational Solution of a H-J-B equation arising from Stochastic Optimal Control Problem

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Abstract

In this paper, we consider numerical solution of a H-J-B (Hamilton-Jacobi-Bellman) equation of elliptic type arising from the stochastic control problem. For the numerical solution of the equation, we take an approach involving contraction mapping and finite difference approximation. We choose the Itô type stochastic differential equation as the dynamic system concerned. The numerical method of solution is validated computationally by using the constructed test case. Map of optimal controls is obtained through the numerical solution process of the equation. We also show how the method applies by taking a simple example of nonlinear spacecraft control.

1. Introduction

Dynamic programming method developed by R. Bellman[1] can be considered as a powerful method of optimal control problem in the sense that it yields necessary and sufficient optimal solutions. Computational solution of the optimality condition, namely the H-J-B equation(Bellman equation), is very difficult due to the complexity and the dimentionality involved in the solution process. In this paper, numerical solution of the Bellman equation arising from the stochastic control problem with infinite time horizon is obtained. In this case, we solve the elliptic type Bellman equation which is a nonlinear partial differential equation. We discretize the Bellman equation applying the finite difference method described in [2], and modified the discrete Bellman equation to the fixed point form using contraction mapping method [3,4] for the iterative numerical solution of the equation. Test case is also constructed in order to validate the numerical scheme. Map of optimal controls is also obtained for both test case and engineering example.

2. The Bellman equation

The Bellman equation also known as the dynamic programming equation arises in the general classes of stochastic control problems such as optimal regulation, tracking and stopping. We take the following stochastic dynamic system which can be either linear or nonlinear.

$$d(y_x^u)_i(s) = m_i^u(s, y_x^u(s))ds + \sum_{j=1}^{n} \sigma_{ij}^u(s, y_x^u(s))dw_j(s) \quad (1)$$

$$i = 1,2,\dots,n$$

where

$y_x^u(0) = x = \{x_1, x_2, \dots, x_n\}$, w_j is a standard Wiener process, and $y_x^u(s)$ represents the solution of (1) at time s evolved from x with control u.

Equation (1) is called the Itô stochastic differential equation if m_i^u and σ_{ij}^u satisfy the so called Itô conditions [5]. We take the following Bolza type of cost functional which can be used for the problem of regulation or tracking. The costs $f^u(\cdot)$ and $\varphi(\cdot)$ become quadratic function in case of regulation or tracking problem, i.e. refer (16).

$$J(x,u)=$$
$$E_{x,u}\{\int_0^{\tau(u)} f^{u(s)}(y_x^{u(s)}(s))\exp\{-\int_0^s c^{u(\sigma)}(y_x^{u(\sigma)}(\sigma))d\sigma\}ds$$
$$+\varphi(y_x^{u(\tau(u))}(\tau(u)))\exp\{-\int_0^{\tau(u)} c^{u(\sigma)}(y_x^{u(\sigma)}(\sigma))d\sigma\}\} \quad (2)$$

where

$E_{x,u}$: conditional expectation for $\{x,u\}$,

$U=\{u_1,u_2,u_3,\ldots\}$: set of all possible control actions,

$u(s)=u_i$,

$f^u(\cdot),\varphi(\cdot)$: costs of random process in the domain (Ω) and boundary $(\partial\Omega)$ of $y_x^{u(s)}$, refer (16) for quadratic case,

$c^{u(\sigma)}$: discount factor,

$\tau(u)=\inf\{t>0, y_x^{u(t)}(t)\notin\Omega\}$, and $E_{x,u}\tau(u)<\infty$ for each $x\in\partial\Omega$.

Let $v(x)=\inf_U J(x,u)$. (3)

Now applying the dynamic programming approach [1] and Itô's lemma[5] to (1),(2),(3) yield the following Bellman equation as the optimality condition, which is an elliptic nonlinear partial differential equation.

$$\max_{u\in U}\{L^u(x)v(x)-f^u(x)\}=0 \text{ for } x\in\Omega \quad (4)$$
$$v(x)=\varphi(x) \text{ for } x\in\partial\Omega$$

where

$$L^u(x)=-\sum_{i,j}a_{ij}^u(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_i b_i^u(x)\frac{\partial}{\partial x_i}+c^u(x),$$
$$a_{ij}^u=\frac{1}{2}\sum_k \sigma_{ik}^u(x)\sigma_{jk}^u(x),\ b_i^u=-m_i^u(x)$$

3. Numerical Solution of the Bellman equation

Finite difference discretization [2] of the Bellman equation yields the following equation.

$$\max_{u\in U}\{\sum_j D_{ij}^u v_j - f_i^u\}=0 \text{ where } i,j=1,2,\ldots,N \quad (5)$$

Modifying the discrete Bellman equation to the form of a fixed point iteration [3,4] yields

$$v_i=\min_{u\in U}\{\sum_j H_{ij}^u v_j+\tilde{f}_i^u\} \quad (6)$$

where

$$H_{ij}^u=\frac{-D_{ij}^u}{D_{ii}^u} \text{ if } i\neq j,\ 0 \text{ otherwise}$$

$$\tilde{f}_i^u=\frac{f_i^u}{D_{ii}^u}$$

4. Construction of a Test Case

The actual performance of an algorithm might be different from what we expected in term of mathematical reasoning. Thus, we construct a test case[6] which can solve the Bellman equation exactly. By comparing the numerical solution with the exact one, the performance of the algorithm can be examined and validated.

For simplicity, we take a 2-D case for the domain of the test case. As a cosequence, let domain Ω be rectangle (Figure 1) such that

$$\Omega=\{(x_1,x_2):0<x_1<\bar{a},0<x_2<\bar{b}\}.$$

In view of the boundary condition of the Bellman equation, the following can be assumed as the exact solution.

$$v(x_1,x_2)=\phi(x_1,x_2)+\lambda\xi(x_1,x_2) \quad (7)$$

where

$$\phi(x_1,x_2)=x_1(x_1-\bar{a})x_2(x_2-\bar{b})$$
$$\xi(x_1,x_2)=\{x_1(x_1-\bar{a})\}^\alpha\{x_2(x_2-\bar{b})\}^\beta$$

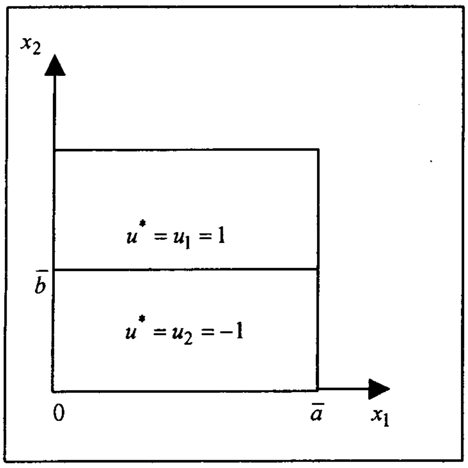

Fig. 1. Domain of the test case

Then $v(x_1, x_2)$ satisfies the boundary condition of the following Bellman equation.

$$\max_{u \in U}\{L^u(x_1,x_2)v(x_1,x_2) - f^u(x_1,x_2)\} = 0 \quad \text{for } (x_1,x_2) \in \Omega \tag{8}$$

$v(x_1,x_2) = 0$ for $(x_1,x_2) \in \partial\Omega$

where

$$L^u(x_1,x_2) = -\sum_{i,j} a_{ij}^u \frac{\partial^2}{\partial x_i \partial x_j} - \sum_i m_i^u \frac{\partial}{\partial x_i} + c^u$$

In order to guarantee the uniqueness of solution[5], we take the operator $L^u(x_1,x_2)$ in the following form.

$$L^u(x_1,x_2) = -(a_{11}^u + \varepsilon)\frac{\partial^2}{\partial x_1^2} - (a_{22}^u + \varepsilon)\frac{\partial^2}{\partial x_2^2} + b_1^u \frac{\partial}{\partial x_1} + b_2^u \frac{\partial}{\partial x_2} + c^u \tag{10}$$

Since $v(x_1,x_2)$ and $L^u(x_1,x_2)$ can be obtained from (7) and (10) respectively, every term in the Bellman equation (8) except $f^u(x_1,x_2)$ is known. Thus if we choose $f^u(x_1,x_2)$ so as to satisfy the equation (8), $v(x_1,x_2)$ become the exact solution of the Bellman equation. Let u^* be the optimal control and take only two values.

$u^* \in U = \{u_1, u_2\}$

Assume

$u^* = u_1$ for $0 < x_2 \le \bar{l}$, and

$u^* = u_2$ for $\bar{l} < x_2 \le b$ (Figure 1).

If the optimal control is u_1, the Bellman equation becomes

$$L^{u_1}(x_1,x_2)v(x_1,x_2) - f^{u_1}(x_1,x_2) = 0 \text{ for } u_1 \tag{11}$$

and

$$L^{u_2}(x_1,x_2)v(x_1,x_2) - f^{u_2}(x_1,x_2) \le 0 \text{ for } u_2 \tag{12}$$

Let $g^u(x_1,x_2) = L^u(x_1,x_2)v(x_1,x_2)$.

Then for (11) and (12),

$f^u(x_1,x_2) = g^{u_1}(x_1,x_2)$,

$f^u(x_1,x_2) = g^{u_2}(x_1,x_2) + \zeta$ for any $\zeta > 0$

For the case that U_2 is the optimal control, $f^u(x_1,x_2)$ can be chosen similarly. As a consequence, $f^u(x_1,x_2)$ can now be determined explicitly using the following expressions.

$$f^u(x_1,x_2) = \begin{cases} g^u(x_1,x_2) \text{ if } (x_2 \le \bar{l} \text{ and } u = u_1) \text{ or } (x_2 > \bar{l} \text{ and } u = u_2) \\ g^u(x_1,x_2) + \zeta \text{ if } (x_2 \le \bar{l} \text{ and } u = u_2) \text{ or } (x_2 > \bar{l} \text{ and } u = u_1) \end{cases}$$

5. Engineering Example

With the construction of a test case, the Bellman equation can be solved not only analytically but also numerically. We take σ_{ij}^u and m_i^u in the following form which represents simple nonlinear attitude dynamics of a spacecraft[7].

$$\sigma_{ij}^u = \begin{bmatrix} 0 & 0 \\ 0 & -\delta_0(\sin x_1 + \eta_0 x_2) \end{bmatrix},$$

$$m_i^u = \begin{bmatrix} x_2 \\ -\sin x_1 + l\sin(2x_1) - \eta_0 x_2 + u \end{bmatrix} \tag{9}$$

The following set Ω is taken as the domain of the example.

$$\Omega = \{(x_1, x_2) : -1 < x_1 < 1, -1 < x_2 < 1\}$$

We take the following quadratic cost functional for the test case.

$$J(x,u) = E_x[\int_0^{\tau(u)} \exp(-cs)\{Y^T(s)QY(s) + ru^2\}ds + \exp\{-c\tau(u)\}Y^T(\tau(u))SY(\tau(u))] \quad (16)$$

where

$$Y(s) = \begin{bmatrix} (y_x^u)_1(s) \\ (y_x^u)_2(s) \end{bmatrix}$$

Take $Q = I = S$ for simplicity.

6. Computational Results

In this section, computational results for both of the test case and the application problem are presented and discussed. The following types of errors are introduced in order to measure performance of the algorithm and also check correctness of the computational results.

Absolute error : $E_{abs}^{\tilde{n}} = \max_i | v_i^{\tilde{n}} - v_i |$

Relative error : $E_{rel}^{\tilde{n}} = \max_i | v_i^{\tilde{n}} - v_i^{\tilde{n}-1} |$

where $v_i^{\tilde{n}}$ and v_i represent the numerical solution at $\tilde{n}_{th}$ iteration and exact solution respectively.

Figure 2 shows the property of contraction mapping, i.e. rapid decrement of the relative error. Figure 3 shows that the absolute error remains constant after certain number of iterations, which we call the steady state error. This gap of error can be reduced if number of grid points taken for finite difference approximation of the operator $L^u(x)$ is increased, and more terms are included in the Taylor series approximation of the operator. Table 1 shows that error decreases as the number of grid points increases. This is reasonable in terms of finite difference approximation that more grid points yield better solution. Figures 4 show the effect of discount factor on the performance of the algorithm, namely bigger discount factor gives smaller absolute error. Figure 5 shows the map of optimal controls obtained for the test case, which is exactly same as the a priori map (refer Fig. 1). Figure 6 shows the map of optimal controls for the example problem. Based on the above discussions, we can conclude that the algorithm gives correct solutions.

7. Conclusions

In this paper, we obtained numerical solution of an elliptic type Bellman equation arising from the stochastic control problem. The Bellman equation of elliptic type is solved by employing the finite difference approximation and contraction mapping method. A test case is constructed in order to solve the Bellman equation not only numerically but also analytically. Consequently, the numerical solution is validated using the test case, i.e. by comparing errors etc. Computational results of the test case show that the algorithm yields reliable solutions. As a result of computational solution, map of optimal controls are obtained for both of the test case and example problem.

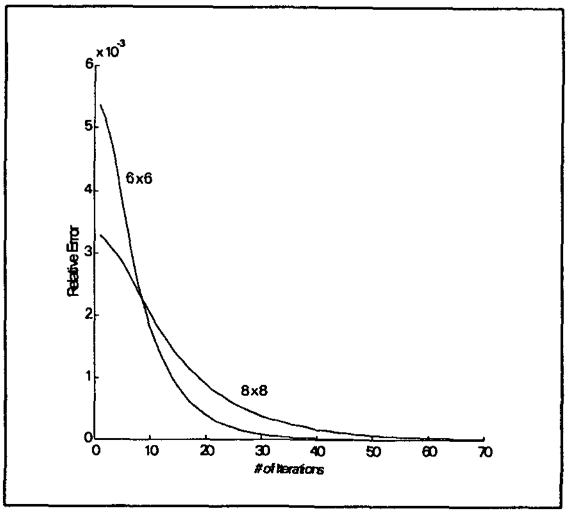

Fig. 2. Relative error vs # of iterations

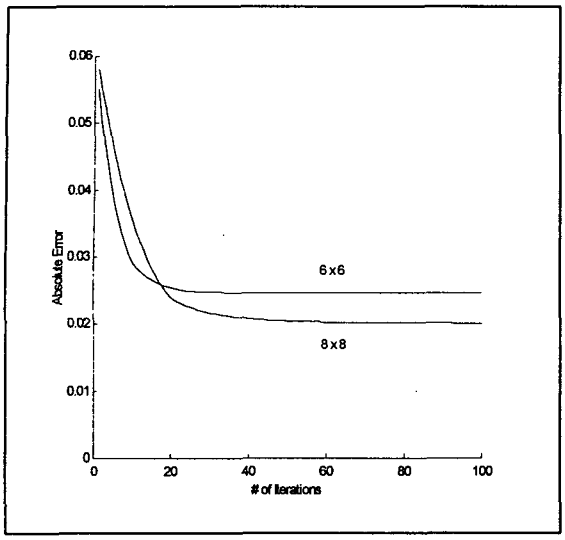

Fig. 3. Absolute error vs # of iterations

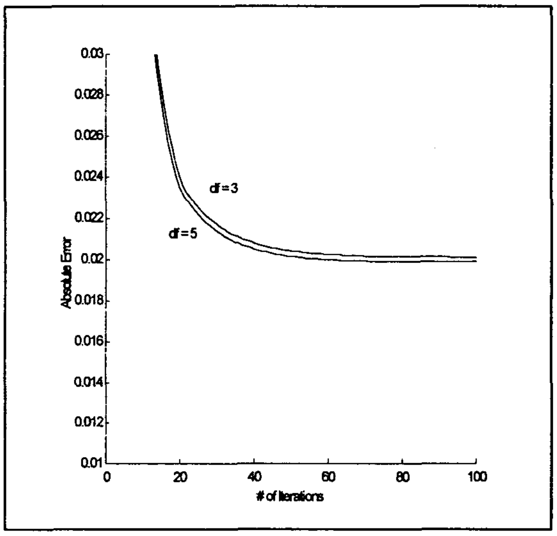

Fig. 4. Effect of discount factors

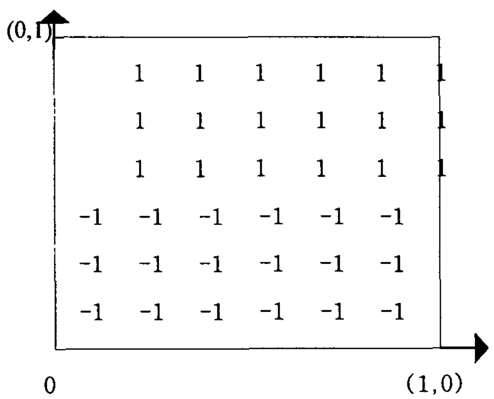

Fig. 5. Map of optimal controls for test case (6 x 6)

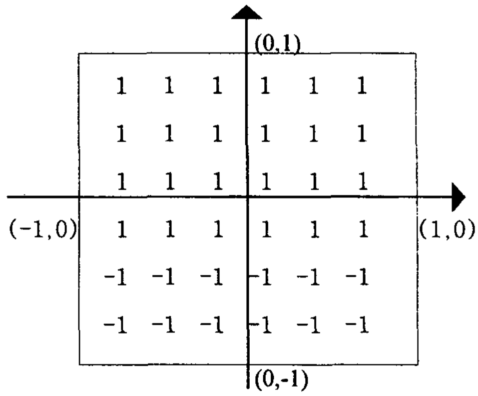

Fig. 6 : Map of optimal controls for example problem

Table 1. Effect of number of grid points

# of Grid Point	Steady State Error (absolute)	Remark
6 x 6	0.2236E-01	
8 x 8	0.1831E-01	

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