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Analysis of the Motion of a Cart with an Inverted Flexible Beam and a Concentrated Tip Mass

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Abstracts In this paper, the mathematical model of a cart with an inverted flexible beam and a concentrated tip mass was derived. The characteristic equation for calculating the natural frequencies of the cart–beam–mass system was obtained and the motion of the system was analyzed through unconstrained modal analysis. A good positioning response of the cart without excessive vibrational motion of the tip mass could be obtained through numerical simulation using PID controller with the feedback of both the position of the cart and the deflection of the beam.

Keywords inverted flexible beam, unconstrained modal analysis, beam vibration, frequency equation, beam deflection

1 Introduction

A lot of research efforts have being done on the analysis and/or control of flexible structures and flexible manipulators. Bhat[1] obtained the exact frequency equations for a uniform cantilever beam carrying a slender tip mass of which center of gravity does not coincide with the attachment point. Anderson[2] obtained the frequency equations for a cantilever with an asymmetrically attached tip mass. Parnell[3] solved the displacement equation for a uniform cantilever beam carrying a concentrated mass at one end using the Laplace transform under generally distributed lateral load and arbitrary boundary and initial conditions. To[4] calculated the natural frequencies and mode shapes of a cantilever beam with a base excitation and tip mass whose center of gravity does not coincide with the point of attachment. Recently, active vibration control problems in mechanically flexible structures such as long beam antennas and slewing flexible structures have received a great deal of attention[5, 6].

All of the above works, however, dealt with a cantilever–type beam carrying a tip mass at one end and the other end of it is fixed at a relatively large inertial frame. When the flexible beam which carries a concentrated tip mass and is fixed on a moving cart is considered, the motion of the beam affects it of the cart, and vice versa. Therefore, the dynamic eqautions and the frequency equation for the cart–beam–mass system must be different with them by above works.

For example, various types of high tower crane systems such as reclaimers in the automatic warehouse and ladder cars which elevate heavy load to high place are used in industrial fields. When they move laterally with heavy load, some vibrational motions due to the flexibility of the main beam are unavoidable, even though they have truss–structured beam, and the vibrational motion is not easily suppressed. Consequently, when a heavy load is moved from a high place to another high place, it is necessary to control the driving motor of the cart such that it generates the torque to suppress the vibration as well as to move the cart to the target position.

In this paper, the mathematical model of an inverted flexible beam which has a concentrated mass at one tip and the other end of which is fixed on the body of a cart was derived. The characteristic equation for calculating the natural frequencies and the exact solution of the motion of the cart–beam–mass system were also obtained through unconstrained modal analysis. Finally, the open–loop response and the response to a simple PID control with and without defelction feedback of the beam of the system were achieved by numerical simulation.

2 Mathematical Modeling

In order to derive equations of motion for the moving cart with a heavy load tipped at a long flexible beam fixed on the cart, the truss–structured beam is simplified as a homogeneous flexible beam[5, 7]. To derive the mathematical model, a cart with an inverted flexible beam and a concentrated tip mass as shown in Fig.1 was considered and the following assumptions were made[8, 9]: 1) The flexible beam is undergoing only plane motion. 2) The deflection of the flexible beam is small and any extension is neglected. 3) The arm is a slender beam with uniform geometric characteristics and homogeneous mass distribution. 4) Rotary

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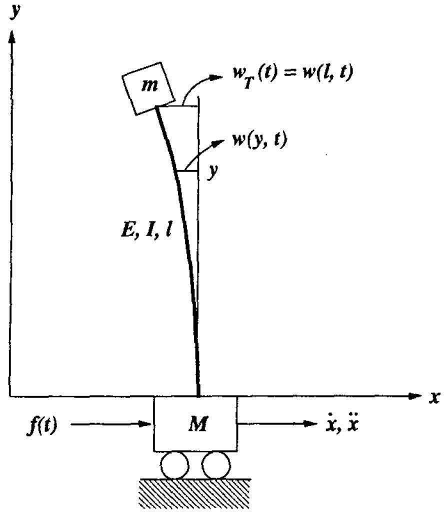

Figure 1: Schematic diagram of a cart with an inverted flexible beam and a concentrated tip mass

inertia and shear deflection effects of the beam are ignored.

The kinetic energy, K_E, and the potential energy, P_E, of the cart and the flexible beam with a concentrated tip mass can be represented as

$$K_E = K_m + \int_0^l K_e dy, \quad P_E = P_m + \int_0^l P_e dy \quad (1)$$

where K_m and P_m are kinetic and potential energy, respectively, due to the motion of rigid masses, and K_e and P_e are the energy density functions of the flexible beam, and are represented, respectively, as follows.

$$K_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[\dot{x} + \dot{w}(l,t)]^2, \quad P_m \approx 0 \quad (2)$$

$$K_e = \frac{1}{2}\rho[\dot{x} + \dot{w}(y,t)]^2, \quad P_e = \frac{1}{2}EI[w''(y,t)]^2 \quad (3)$$

where M is the mass of the cart, m is the mass of the concentrated mass at tip, ρ is the mass per unit length of the beam, EI is the flexural rigidity of the beam and $w(y,t)$ is the deflection of the beam at y.

The following equation is obtained from Hamilton's principle.

$$\int_{t_1}^{t_2} \left\{ \delta L_m + \int_0^l \delta L_e dy + \delta W_{nc}(t) \right\} dt = 0 \quad (4)$$

where $L_m = K_m - P_m$, $L_e = K_e - P_e$, t_1 and t_2 are arbitrary time and $\delta W_{nc}(t) = f(t)\delta x$ where $f(t)$ is applied force and W_{nc} is the work done by nonconsevative forces. From eqs(2) and eqs(3), it can be known that

$$L_m = L_m(\dot{x}, \dot{w}_T), \quad L_e = L_e(\dot{x}, \dot{w}, w'') \quad (5)$$

where $w_T(t) = w(l,t)$, $w' = \partial w/\partial y$ and $\dot{w} = \partial w/\partial t$.

Substituting the variations of eqs(5) into eq(4), integrating the resulting equation by parts, and considering that the time t_1 and t_2 are arbitrary and that δx and δw are arbitrary and independent, the following equations of motion and boundary conditions are obtained.

$$M_t\ddot{x} + m\ddot{w}_T + \rho \int_0^l \ddot{w}(y,t)dy = f(t) \quad (6)$$

$$EIw''''(y,t) + \rho[\ddot{x} + \ddot{w}(y,t)] = 0 \quad (7)$$

$$EIw'''(l,t) = m(\ddot{x} + \ddot{w}_T), \quad w''(l,t) = 0 \quad (8)$$

$$w(0,t) = 0, \quad w'(0,t) = 0 \quad (9)$$

where $M_t = M + m + m_b$ is the total mass of the system, $m_b = \rho \int_0^l dy$ is the mass of the flexible beam. It can be known that eqs(8) are natural bondary conditions and eqs(9) are geometrical ones.

3 Unconstrained Modal Analysis

3.1 Exact Modal Analysis

The exact modal analysis of eq(6) and eq(7) is accomplished without forcing $\ddot{x}(t)$ to zero[9, 10]. Thus, the position of the cart, $x(t)$, is assumed to have a solution of the form

$$x(t) = \alpha(t) + \beta v(t) \quad (10)$$

where $\alpha(t)$ describes the motion of the link center of mass, and $w(y,t)$ is assumed as

$$w(y,t) = \phi(y)v(t) \quad (11)$$

Substituting eq(10) and eq(11) into eq(6) and eq(7) yields, respectively

$$M_t\ddot{\alpha}(t) + \left[\beta M + m\psi(l) + \rho \int_0^l \psi(y)dy\right] \ddot{v}(t) = f(t) \quad (12)$$

and

$$EIv(t)\psi''''(y) + \rho\psi(y)\ddot{v}(t) = -\rho\ddot{\alpha}(t) \quad (13)$$

where $\psi(y) = \beta + \phi(y)$. In eq(12), if β satisfies

$$\beta = -\frac{m}{M}\psi(l) - \frac{\rho}{M}\int_0^l \psi(y)dy \quad (14)$$

then the motion of center of mass without perturbation is obtained as

$$M_t\ddot{\alpha}(t) = f(t) \quad (15)$$

To get the normal mode solution for the vibrating beam, by letting $\alpha(t) = 0$, eq(13) is decomposed as:

$$\ddot{v}(t) + \omega^2 v(t) = 0 \quad (16)$$

$$\psi''''(y) - k^4\psi(y) = 0 \quad (17)$$

where $k^4 = \rho\omega^2/EI$. The boundary conditions for $\psi(y)$ are given as follows.

$$\psi(0) = \beta, \quad \psi'(0) = 0 \quad (18)$$

$$\psi''(l) = 0, \quad \psi'''(l) = -k^4\frac{m}{\rho}\psi(l) \quad (19)$$

Table 1: First 10 roots of the frequency equations

i	by the frequency equation (21) $\frac{m}{M}=0$	0.1	0.5	1	2	by the frequency equation (22) $\frac{m}{M}=0$	0.1	0.5	1	2	by eq(23)
1	1.8846	1.0123	0.7413	0.6702	0.6234	1.8751	0.9849	0.6688	0.5635	0.4743	1.8751
2	4.7015	3.9738	3.9432	3.9392	3.9372	4.6941	3.9654	3.9347	3.9307	3.9286	4.6941
3	7.8590	7.0967	7.0781	7.0757	7.0745	7.8548	7.0920	7.0734	7.0710	7.0698	7.8548
4	10.9986	10.2298	10.2168	10.2151	10.2143	10.9955	10.2265	10.2135	10.2118	10.2110	10.9955
5	14.1396	13.3668	13.3568	13.3556	13.3549	14.1372	13.3643	13.3543	13.3530	13.3524	14.1372
6	17.2807	16.5056	16.4975	16.4964	16.4959	17.2788	16.5035	16.4954	16.4944	16.4939	17.2788
7	20.4220	19.6452	19.6384	19.6375	19.6371	20.4204	19.6435	19.6367	19.6358	19.6354	20.4204
8	23.5634	22.7854	22.7795	22.7788	22.7784	23.5619	22.7839	22.7780	22.7773	22.7769	23.5619
9	26.7048	25.9259	25.9207	25.9201	25.9198	26.7035	25.9246	25.9194	25.9188	25.9185	26.7035
10	29.8463	29.0667	29.0621	29.0615	29.0612	29.8451	29.0655	29.0609	29.0603	29.0600	29.8451

The general sloution to eq(17) has the form:

$$\psi(y) = A\cos ky + B\sin ky + C\cosh ky + D\sinh ky \tag{20}$$

Solving the eigenvalue problem in eq(17)–eq(19) with eq(14), the frequency equation is obtained as follows.

$$\begin{aligned} &1+\cos\xi_i\cosh\xi_i + 2r_1\cos\xi_i\cosh\xi_i \\ &+\frac{r_2}{\xi_i}(\cos\xi_i\sinh\xi_i+\sin\xi_i\cosh\xi_i) \\ &+r_3\xi_i(\cos\xi_i\sinh\xi_i-\sin\xi_i\cosh\xi_i)=0 \end{aligned} \tag{21}$$

where $r_1 = m/M$, $r_2 = m_b/M$, $r_3 = m/m_b$, $\xi = kl$.

If eq(7) is solved by *constrained modal analysis* by letting $\ddot{x}(t) = 0$, then the resulting frequency equation has the form:

$$\begin{aligned} &1+\cos\xi_i\cosh\xi_i \\ &+ r_3\xi_i(\cos\xi_i\sinh\xi_i-\sin\xi_i\cosh\xi_i)=0 \end{aligned} \tag{22}$$

In addition, it can be also known that if one end of a flexible beam without concentrated mass is fixed at a large inertial frame, *i.e.*, if $M \to \infty$ and $m = 0$, then $r_1 \to 0$, $r_2 \to 0$, and $r_3 = 0$, and the frequency equation becomes as follows[10].

$$1+\cos\xi_i\cosh\xi_i = 0 \tag{23}$$

The roots of the frequency equations by both the *unconstrained* and the *constrained* modal analysis for several mass ratio are compared with the results by eq(23) for the values of $M = 10kg$, $m_b = 0.33875kg$ and $l = 1.0m$, and Table 1 lists the first 10 roots of the equations. When the ratio between the mass of the cart and the concentrated tip mass is sufficiently small, the eigenvalues by eq(22) approximate the exact values by eq(21) with good accuracy. However, as the mass ratio increases, the difference of the eigenvalues between the two eqautions become more significant, especially near the first root.

Fig. 2 shows the change of the fundamental frequency, $\omega_1 = (EIk_1^4/\rho)^{1/2}$, for given $k_1 = \xi_1/l$ by eq(21), eq(22) and eq(23) with respect to the change of the mass ratio when the system parameters are as shown in Table 2. The

Table 2: System parameters

Base mass, M	10.0 kg
Beam length, l	1.0 m
Mass per unit length, ρ	0.33875 kg/m
Young's modulus, E	7.1 $\times 10^{10}$ N/m^2
Mass moment of inertia, I	6.51 $\times 10^{-11}$ kg-m^2

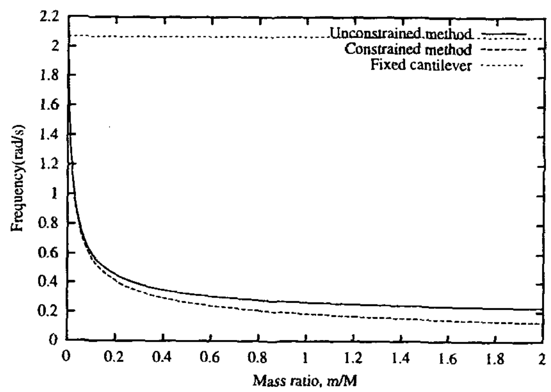

Figure 2: The fundamental frequency vs. the mass ratio

frequencies for the characteristic values obtained by eq(23), the frequency equation for the fixed cantilever beam without concentrated mass, greatly differ from them by both eq(21) and eq(22) except $m/M = 0$.

For given k_i, the mode shape, $\phi_i(y) = \psi_i(y) - \beta_i$, can be obtained by eq(20). The constants $A_i, \cdots, D_i$ and β_i are determined by solving eq(20) and eq(14) simultaneously with the boundary conditions and

$$\beta_i = \frac{\rho}{Mk_i^4}\psi'''(0) = -\frac{2\rho B_i}{Mk_i} \tag{24}$$

Fig. 3 shows the first three normalized modes with respect to the parameters in Table 2.

For given $\psi_i(y)$ and β_i, the beam deflection is given as

$$w(y,t) = \sum_{i=1}^{\infty} v_i(t)\left(-\beta_i + \psi_i(y)\right) \tag{25}$$

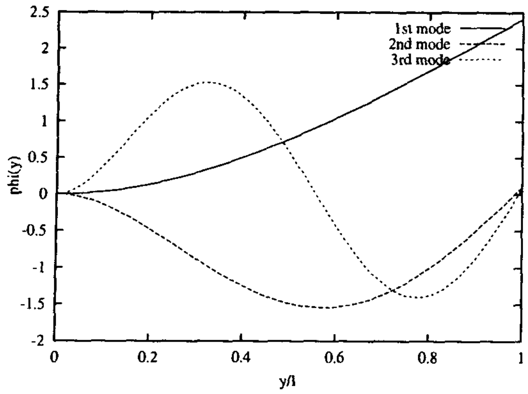

Figure 3: The shape of the first three modes

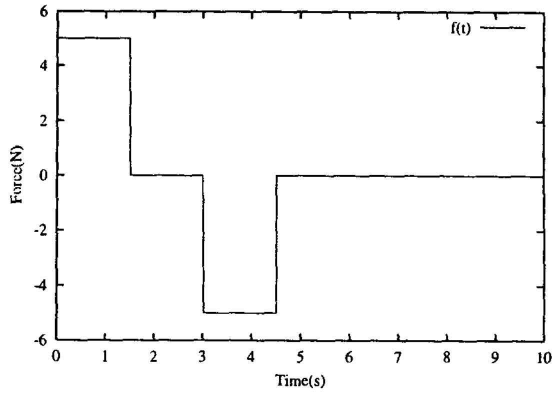

Figure 4: An arbitrary forcing function

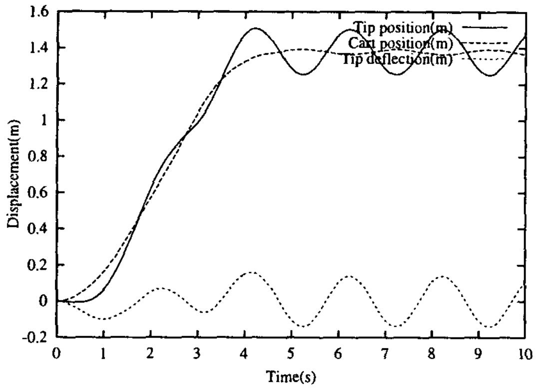

Figure 5: The open-loop response to an arbitrary forcing function

and accordingly, the cart position is obtained as

$$x(t) = \alpha(t) + \sum_{i=1}^{\infty} \beta_i v_i(t) \tag{26}$$

From the condition of eq(17), eq(13) is rewritten as

$$\sum_{i=1}^{\infty} \psi_i(y) \left[\ddot{v}_i(t) + \omega_i^2 v_i(t)\right] = -\ddot{\alpha}(t) \tag{27}$$

Multiplying both sides of eq(27) by $\psi_j(y)$, integrating over the problem domain, and applying the orthogonality property and eq(14), it becomes

$$\ddot{v}_i(t) + \omega_i^2 v_i(t) = \frac{f(t)}{M_t \rho}\left[m\psi_i(l) + M\beta_i\right] \tag{28}$$

for $i = 1, 2, \cdots, \infty$. Then, $v_i(t)$ are obtained by integrating the eq(28) for given ω_i.

3.2 Finite–Dimensional Solutions

Finite–dimensional approximated models for a finite number of eigenvalues/eigenvectors are obtained from the previous development by considering the first n roots of the characteristic equation, eq(21). The link deflection and the displacement of the cart is expressed, respectively, in terms of n mode shapes as follows.

$$w(y,t) = \sum_{i=1}^{n} v_i(t)\left(\psi_i(y) - \beta_i\right) \tag{29}$$

$$x(t) = \alpha(t) + \sum_{i=1}^{n} \beta_i v_i(t) \tag{30}$$

Now, the inhomogeneous equations eq(6)–eq(9) can be transformed into a set of $n+1$ second-order ordinary differential equations of the form:

$$\begin{pmatrix} M_t & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \ddot{\alpha} \\ \ddot{v}_i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} \alpha \\ v_i \end{pmatrix} = \begin{pmatrix} 1 \\ \zeta_i \end{pmatrix} f(t) \tag{31}$$

for $i = 1, \cdots, n$, where $K = \text{diag}\{\omega_i^2\}$ is an $n \times n$ *stiffness matrix* and $\zeta_i = [M\beta_i + m\psi_i(l)]/(M_t\rho)$.

4 Numerical Simulations

4.1 Open–Loop Response

Numerical simulations were carried out to examine the open–loop response of the system to the arbitrarily designed forcing functions. The parameters listed in Table 2 were used in the simulation. Fig. 5 shows the position of the cart and the tip mass when the forcing function shown in Fig. 4 is applied at the condition of $m = 1kg$ and $n = 3$. The tip position, $x_T(t)$, is given by

$$x_T(t) = x(t) + w_T(t) \tag{32}$$

As shown in Fig. 5, the vibration generated at the beam is not damped out because the beam is modeled as an elastic one without damping, and the vibrational motion of the tip mass affects the motion of the cart.

4.2 Simple PID Control Using only the Position Feedback of the Cart

The response to a simple PID controller using only the position feedback of the cart was investigated by numerical simulations. The actuating force, $f(t)$, by the controller is

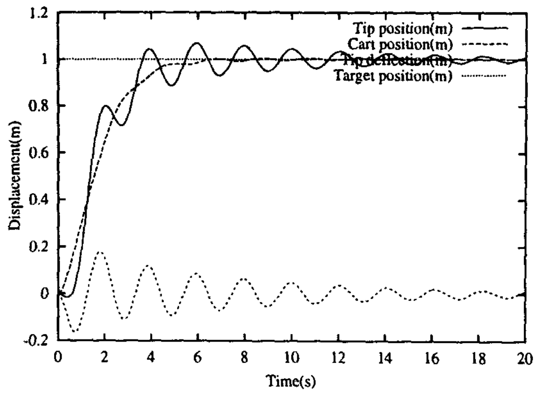

Figure 6: Step response to a simple PID control using only the position feedback of the cart

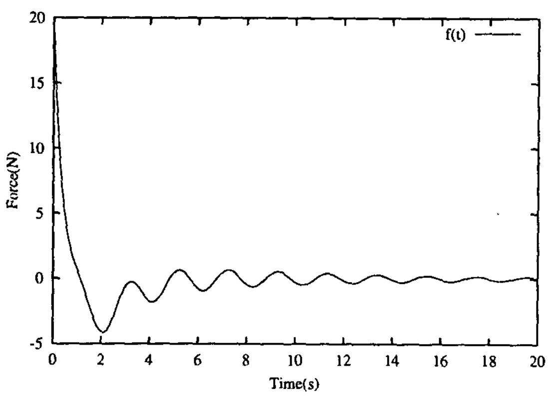

Figure 7: Forcing function by a simple PID control using only the position of the cart

given as follows.

$$f(t) = K_p e(t) + K_v \dot{e}(t) + K_i \int e(t)dt \qquad (33)$$

where $e(t) = x_d(t) - x(t)$. Fig. 6 shows the response of the cart and the tip mass to a step input when $K_P = 20$, $K_v = 30$, $K_i = 0$, $x_d = 1m$ and $n = 3$. Fig. 7 shows the actuating force calculated by eq(33).

As shown in Fig. 6, the vibrational motion goes down as time passes. The result is caused by the fact that the position of the cart, $x(t)$, includes both the rigid body motion, $\alpha(t)$, and the sinusoidal time-varying function, $v(t)$, as in eq(30). Although the simple PID controller using only the position feedback of the cart does not destabilize the system, the system performance is not satisfactory because the flexible modes of the beam are seriously excited and not effectively suppressed.[11]

In Fig. 6, it can be known that the tip position, x_T, stay behind the initial position. This problem is caused by the fact that the tip defelction is obtained by integrating the mode shape, $\psi_i(y)$, from 0 to l despite that the tip deflection accompanies so-called the *shortening effect*.

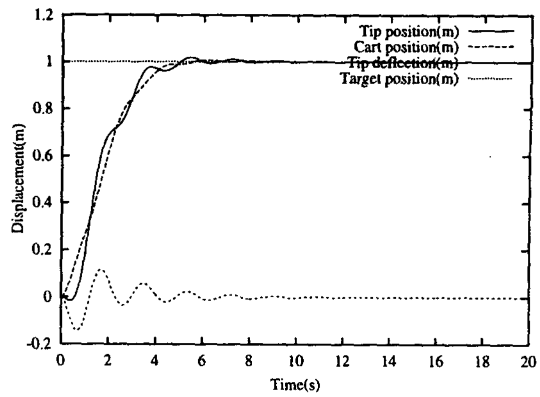

Figure 8: Step response to a PID control using both the feedback of the cart position and the tip deflection

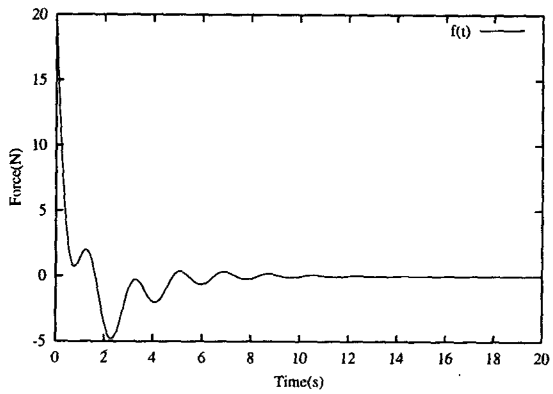

Figure 9: Forcing function by the PID control using both the cart position and the tip deflection

4.3 PID Control Using both the Position of the Cart and the Deflection of the Beam

In the first place, only the tip deflection of the flexible beam was fedback and added to the simple PID control action to reduce the vibrational motion during the transient response. The actuating force by this controller is given as follows.

$$f(t) = K_p e(t) + K_v \dot{e}(t) + K_i \int e(t)dt + K_{pw} w_T(t) \quad (34)$$

Fig. 8 shows the response of the cart and the tip mass when $K_P = 20$, $K_v = 30$, $K_i = 0$, $K_{pw} = 30$ and $n = 3$. Fig. 8 shows that the vibrational motion at the tip is reduced compared with the response to the simple PID control although the tip deflection is still not small. Fig. 9 shows the actuating force by eq(34).

For the second time, feedback of the time derivative of the tip deflection as well as the tip deflection was added to the simple PID controller to generate the control force as follows.

$$\begin{aligned} f(t) = & K_p e(t) + K_v \dot{e}(t) + K_i \int e(t)dt \\ & + K_{pw} w_T(t) + K_{vw} \dot{w}_T(t) \end{aligned} \qquad (35)$$

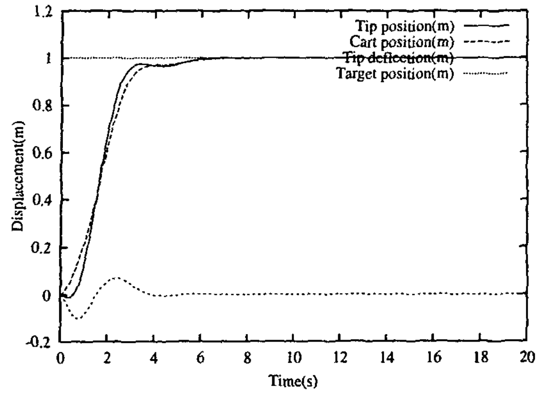

Figure 10: Step response to a PID control using the cart position, the tip deflection and the derivative of it

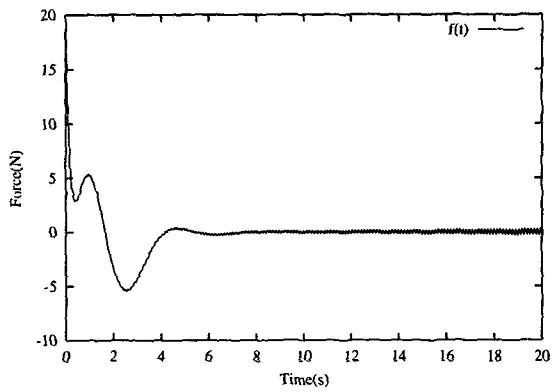

Figure 11: Forcing function by the PID control using the cart position, the tip deflection and the derivative of it

Fig. 10 shows the response of the cart and the tip mass when $K_P = 20$, $K_v = 30$, $K_i = 0$, $K_{pw} = 30$, $K_{vw} = 30$ and $n = 3$. In Fig. 10, it can be known that the vibrational motion and the deflection at the tip are remarkably reduced compared with the response to the previous controllers.

Fig. 11 is the actuating force by eq(35). The result shows the possibility of divergence of the response because of the diverging input force by the high frequency modes. Therefore, from the view point of practical implementation, the controller using the derivative of the tip deflection can generate unstable motion due to the high frequency modes.

5 Conclusions and Future Works

In this paper, the equations of motion and the boundary conditions of a cart with an inverted flexible beam and a concentrated tip mass was derived. The frequency equation for the cart–mass–beam system and the exact and finite-dimensional mode solutions were obtained by unconstrained modal analysis. The open-loop response by an arbitrary forcing function and the response to both the PID controllers using the position feedback of the cart with and without the feedback of the deflection at the tip of the flexible beam were obatined by numerical simulations.

As the future work, the experimental results on a test bed will be compared with the simulated ones and the controller with good performance will be designed. The mathematical model and control algorithm will also be developed for a cart with an inverted flexible beam and a moving mass.

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