Proceedings of the 13th
KACC, October 1998

Identification Using Orthonormal Functions

°Chul-Min BAE*, Kiyoshi WADA*, Jun IMAI*

*Department of Electrical and Electronic Systems Engineering, Kyushu University
Fukuoka, 812-8581 JAPAN
Tel: +81-92-642-3958; Fax:+81-92-642-3939;
E-mail:bae@dickie.ees.kyushu-u.ac.jp

Abstract A least-squares identification method is studied that estimates a finite number of coefficients in the series expansion of a transfer function, where the expansion is in terms of recently introduced generalized basis functions. We will expand and generalize the orthogonal functions as basis functions for dynamical system representations. To this end, use is made of balanced realizations as inner transfer functions. The orthogonal funcitions can be considered as generalizations of, for example, the pulse functions, Laguerre functions, and Kautz functions, and give rise to an alternative series expansion of rational transfer functions. We show that the Laplace transform of the expansion for some sets $\Psi_k(z)$ is equivalent to a series expansion. Techniques based on this result are presented for obtaining the coefficients c_n as those of a series. One of their important properties is that, if chosen properly, they can substantially increase the speed of convergence of the series expansion. This leads to accurate approximate models with only a few coefficients to be estimated. The set of Kautz functions is discussed in detail and, using the power-series equivalence, the truncation error is obtained.

Keywords Laguerre functions, Kautz functions, Dynamical system, identification

1 Introduction

The use of orthogonal basis functions for the Hilbert space of stable systems has a long history in the modelling and identification of dynamical systems. The main part of this work dates back to the classical work of Lee(1933) and Wiener(1949), which is summarized in Lee(1960).
Given the fact that every stable system has a unique series expansion in terms of a pre-chosen basis, a model representation in terms of a finite-length series expansion can serve as an approximate model, where the coefficients of the series expansion can be estimated from input-output data. Consider for example a stable system $G(z)$, written as

$$G(z) = \sum_{k=1}^{\infty} g_k z^{-k} \tag{1}$$

with $\{g_k\}_{k=0,1,2,\cdots}$ the sequence of Markov parameters. Let $\{f_k(z)\}_{k=0,1,2,\cdots}$ be an orthonormal basis for the set of systems. Then there exists a unique series expansion

$$G(z) = \sum_{k=1}^{\infty} w_k f_k(z), \tag{2}$$

with $\{w_k\}_{k=1,2,\cdots}$ the expansion coefficients. The orthonormality of the basis is reflected by the property that

$$\int_a^b f_m(z) f_n(z) dz = \begin{cases} 1, & if \quad m = n, \\ 0, & if \quad m \neq n. \end{cases} \tag{3}$$

When the basis functions satisfy the first condition, they are said to be normalized; and, when they satisfy the second condition, they are said to be orthogonal. A model of the system $G(z)$ can be approximated by a finite-length series expansion

$$\widehat{G}(z) = \sum_{k=1}^{n-1} \widehat{w}_k f_k(z) \tag{4}$$

where the accuracy of the model will be essentially dependent on the choice of basis functions $f_k(z)$. Note that the choice $f_k(z) = z^{-k}$ corresponds to the use of so-called FIR (finite impulse response) models. Since the accuracy of the models is limited by the basis functions.

2 Kautz Function

The problem of orthogonalizing a set of continuous time exponential functions has been elegantly solved in [1]. The key idea is to determine the corresponding Laplace transforms, which have very simple structures.
The sequence of functions $\{\Psi_k(z)\}$ is determined as follows

$$\Psi_{2k-1}(z) = C_1^{(k)}(1 - a_1^{(k)} z)\Gamma^{(k)}(z) \tag{5}$$

$$\Psi_{2k}(z) = C_2^{(k)}(1 - a_2^{(k)} z)\Gamma^{(k)}(z) \tag{6}$$

$$\Gamma^{(k)}(z) = \frac{\prod_{j=1}^{k-1}(1-\beta_j z)(1-\beta_j^* z)}{\prod_{j=1}^{k}(z-\beta_j)(z-\beta_j^*)}$$

$$C_1^{(k)} = \sqrt{\frac{(1-\beta_k^2)(1-\beta_k^{*2})(1-\beta_k\beta_k^*)}{(1+(a_1^{(k)})^2)(1+\beta_k\beta_k^*) - 2a_1^{(k)}(\beta_k+\beta_k^*)}}$$

$$C_2^{(k)} = \sqrt{\frac{(1-\beta_k^2)(1-\beta_k^{*2})(1-\beta_k\beta_k^*)}{(1+(a_2^{(k)})^2)(1+\beta_k\beta_k^*) - 2a_2^{(k)}(\beta_k+\beta_k^*)}}$$

$$(1+a_1^{(k)}a_2^{(k)})(1+\beta_k\beta_k^*) - (a_1^{(k)}+a_2^{(k)})(\beta_k+\beta_k^*) = 0 \tag{7}$$

Here $\beta_k's$ are complex numbers such that $|\beta_k| < 1$, and $a_1^{(k)}$, $a_2^{(k)}$ are restricted by the condition (7). The functions $\{\Psi_k(z)\}_{k=1,2,\cdots}$ will be called the discrete Kautz functions. Another special case is for $\beta_k = \beta$. For this case one can take

$$a_1^{(k)} = \frac{1+\beta\beta^*}{\beta+\beta^*}, \quad a_2^{(k)} = 0$$

and thus

$$\begin{aligned}\Psi_{2k-1}(z) &= \frac{\sqrt{1-c^2}(z-b)}{z^2+b(c-1)z-c} \left[\frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}\right]^{k-1} \\ &= K_{2k-1}(z)G_b(z)^k\end{aligned}$$

$$\begin{aligned}\Psi_{2k}(z) &= \frac{\sqrt{(1-c^2)(1-b^2)}}{z^2+b(c-1)z-c} \left[\frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}\right]^{k-1} \\ &= K_{2k}(z)G_b(z)^k\end{aligned}$$

$$|b| < 1, \quad |c| < 1$$

where

$$\begin{aligned}K_{2k-1}(z) &= \frac{\sqrt{1-c^2}(z-b)}{z^2+b(c-1)z-c} \\ K_{2k}(z) &= \frac{\sqrt{(1-c^2)(1-b^2)}}{z^2+b(c-1)z-c} \\ G_b(z) &= \frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}\end{aligned}$$

and $b = (\beta+\beta^*)/(1+\beta\beta^*)$, $c = -\beta\beta^*$. Since $a_1^{(k)}$ and $a_2^{(k)}$ are not unique, several other sets of $\{\Psi_k(z)\}$ are possible.
Denote

$$\begin{aligned}V_k(z) &= \begin{bmatrix} K_{2k-1}(z) \\ K_{2k}(z) \end{bmatrix} G_b(z)^k \\ &= \frac{\sqrt{1-c^2}}{z^2+b(c-1)z-c}\begin{bmatrix} z-b \\ \sqrt{1-b^2} \end{bmatrix} G_b(z)^k\end{aligned}$$

3 Identification of expansion coefficients

Using Laguerre and Kautz function a practical parameter identification method for linear time-invariant systems is introduced. System identification deals with the problem of finding an estimate of $G(z)$ from observations of $\{y(t), u(t)\}_{t=1\cdots N}$. The identification problem simplifies to a linear regression estimation problem if the model is linear-in-the-parameters, and can be represented by

$$G(z) = \sum_{k=1}^{n} w_k f_k(z) \tag{8}$$

where $\{f_k(z)\}$ is a set of given basis functions and $\{w_k\}$ are the unknown model parameters. If $\left\{\begin{bmatrix} f_{2k-1}(z) \\ f_{2k}(z) \end{bmatrix}\right\}$ correspond to $\{V_k(z)\}$, we call this model a Kautz model. The least squares method can now be applied to estimate the model parameters

$$\theta^T = (w_1, w_2, \cdots, w_n) \quad . \tag{9}$$

The input/output relation can be written in the linear regression form

$$y(t) = z_t^T \theta \tag{10}$$

where

$$z_t^T = [\bar{u}_1(t), \bar{u}_2(t), \cdots, \bar{u}_n(t)]$$

$$\bar{u}_k(t) = f_k(z)u(t),$$

Let

$$Z^T = [z_{t0}, \cdots, z_N], \quad \mathbf{y}^T = [y(t0), \cdots, y(N)]$$

Then, the least squares estimate of θ minimizes the loss function

$$\begin{aligned} J &= \frac{1}{N}\sum_{t=t_0}^{N}(y(t) - z_t^T\theta)^2 \\ &= \frac{1}{N}(\mathbf{y} - Z\theta)^T(\mathbf{y} - Z\theta) \end{aligned} \qquad (11)$$

The solution of this quadratic optimization problem is

$$\hat{\theta}_N = (Z^T Z)^{-1} Z^T \mathbf{y} \qquad (12)$$

where

$$\begin{aligned} Z^T Z &= \frac{1}{N}\sum_{t=t_0}^{N} z_t z_t^T, \\ Z^T \mathbf{y} &= \frac{1}{N}\sum_{t=t_0}^{N} z_t y(t). \end{aligned} \qquad (13)$$

The value of t_0 depends on how the effects of unknown initial conditions are treated. For large N, the effects of t_0 will be negligible.

4 Simulation Example

We give a simple example to illustrate the advantage of using Kautz models for second order resonant systems. Consider a continuous time transfer function

$$G^0(s) = \frac{1}{s^2 + 0.2s + 1} \qquad (14)$$

with resonant frequency $\omega_0 = 1$ and damping 0.1. This system is sampled using a zero-order hold with sampling period $T = 0.5$. The FIR approximation is shown in Fig. 1, and 2. The second-order Kautz model approximation, $n = 2$, is given in Fig.3. The Kautz approximation of order 7 in shown in Fig.4.

5 Conclusion

In this paper we considered an estimation method of transfer function $G(z)$ using basis functions expansion. We illustrated by numerical example that the presented method of identification can be performed with good accuracy using a rather smaller numbers of expansion terms than that for the case where the FIR model is used.

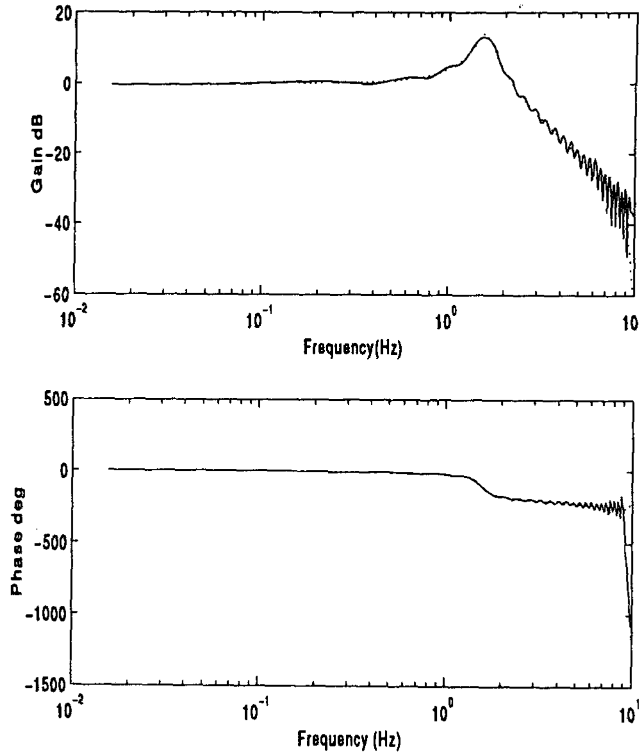

Figure 1 : Solid line-true system,dashed line-FIR model of order 50

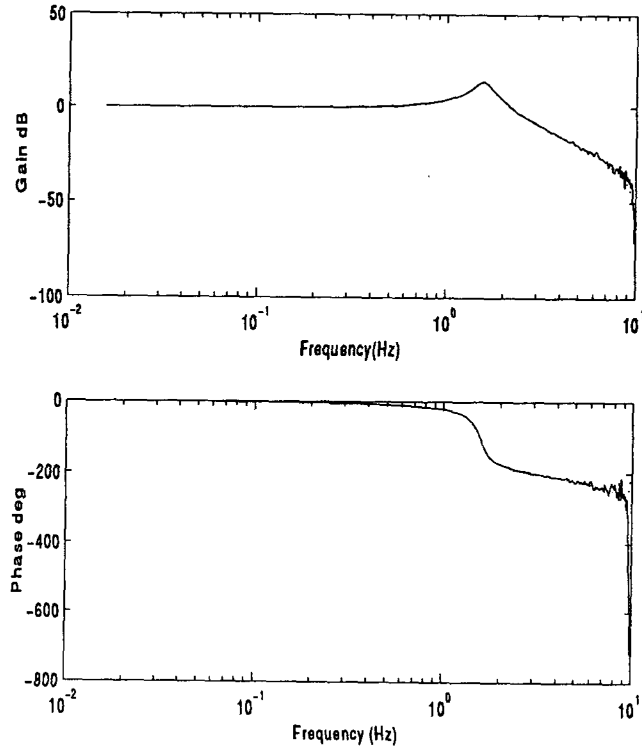

Figure 2 : Solid line-true system,dashed line-FIR model of order 100

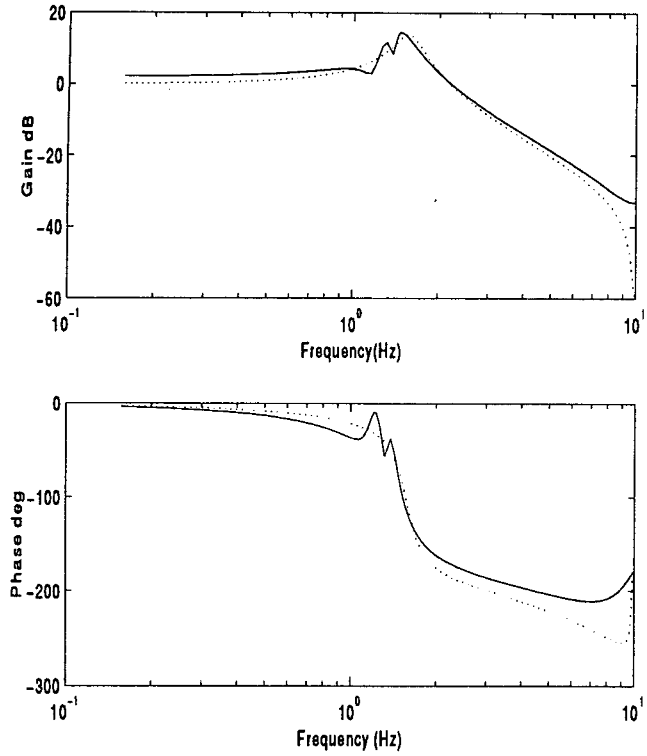

Figure 3 : Solid line-true system,dashed line-Kautz model of order 2

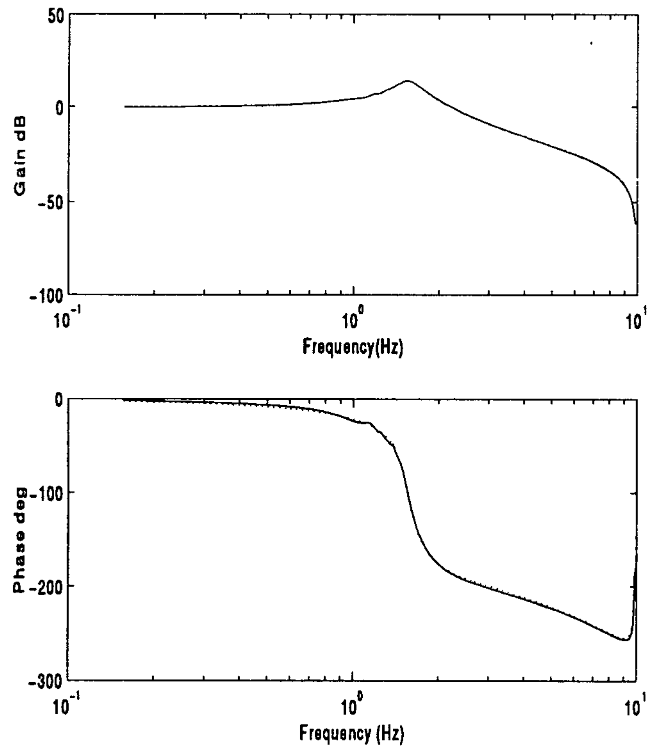

Figure 3 : Solid line-true system,dashed line-Kautz model of order 7

References

[1] Paul M. J. Van den Hof, Peter S.C. Heuberger and Jozsef Bokor "System identification with generalized orthonormal basis functions, " *Automatica*, vol. 31, no. 12, pp. 1821–1834, 1995.

[2] Chyi Hwang and Yen-ping Shih, "Parameter identification via Laguerre polynomials, " *Int. J. Systems Sci.*, vol. 13, no. 2, pp. 209–217, 1982.

[3] Bo Wahlberg, "System identification using Laguerre models, " *IEEE Trans. Autom. Control*, vol. 36 no. 5, pp. 551–562, 1991.

[4] Bo Wahlberg, "System identification using Kautz models, " *IEEE Trans. Autom. Control*, vol. 39, no. 6, pp.1276–1282, 1994.

[5] Y. Fu and G. A. Dumont, "An optimum time scale for discrete Laguerre network, " *IEEE Trans. Autom. Control*, vol. 38, no. 6, pp. 934–938, 1993.

[6] P. R. Clement, "Laguerre functions in signal analysis and parameter identification, " *J. Franklin Inst*, vol. 313, no. 2, pp. 85–95, 1982.

[7] E. J. Hannan and B. Wahlberg, "Convergence rates for inverse Toeplitz matrix forms, " *J. Multivariate Analysis*, vol. 31, pp. 127–135, 1989.

[8] Y. W. Lee, "Choice of the time-scaling factor for linea system approximations using orthonormal Laguerre functions, " *IEEE Trans. Autom. Control*, pp. 487–489, 1965.

[9] Peter S. C. Heuberger, Paul M.J. Van den Hof and Okko H. Bosgra, "A generalized orthonormal basis for linear dynamical systems, " *IEEE Trans. Autom. Contol*, vol. 40, no. 3 pp. 451–465, 1995.

[10] Y. W. Lee "Statistical theory of communication", *New York: Wiley*, 1960.