Realization of Robust Performance for Interval Systems Using Model Reference Feedback

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Abstract

The physical parameters of controlled systems are uncertain and are accompanied with nonlinearity. The transfer function of the controlled system should, therefore, be expressed by interval polynomials. This paper describes the realization of robust performance for that type of control systems (interval systems) via model reference feedback. First, we will analyze an invariance problem of dynamic characteristics such that the dominant roots do not break away from a specified circular area, and will present a discrimination algorithm (i.e., a division algorithm) for the extreme points of the uncertain coefficients. Then, we will present a design method of control systems which have a robust performance such that the location of the dominant roots does not vary excessively.

1. Introduction

Since the physical parameters of controlled systems are uncertain and are accompanied with nonlinearity, the transfer function should be expressed by interval polynomials [1, 2]. This paper describes the existing area of characteristic roots for control systems which are expressed by that type of transfer function. A discrimination method of the number of characteristic roots in a specified area on an s-plane was developed in our paper [3], when a characteristic equation was expressed as an interval polynomial. The criterion is based on the classic Sturm's theorem. The discrimination algorithm was expressed so that it can be easily programmed on a computer. This paper examines an invariance problem of dynamic characteristics such that the dominant roots do not break away from a specified circular area (a disc), and presents a discrimination algorithm (i.e., a division algorithm) for the extreme points of the uncertain coefficients. Designing examples of a control system with robust performance using model reference feedback are presented.

2. Interval Polynomials

The transfer function of a control system with uncertainty (and nonlinearity) is expressed by interval polynomials.
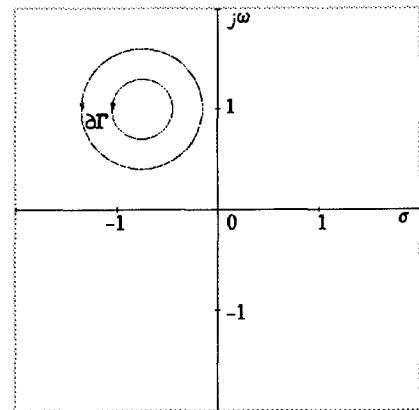

Fig. 1 Circular area ($\sigma_0 = -0.75$ and $\omega_0 = 1.0$, $\rho = 0.3$ or $\rho = 0.6$.

Therefore, the characteristic polynomial of a control system with uncertainties can be written by an interval polynomial as follows:

$$\check{F}(s) = \check{a}_0 s^n + \check{a}_1 s^{n-1} + \cdots + \check{a}_{n-1} s + \check{a}_n, \quad (1)$$
$$\check{a}_i \in [a_i^-; a_i^+], \quad (i = 0, 1, 2, \cdots, n).$$

When expressing interval polynomials in terms of nominal coefficient a_i and maximum error $\Delta \bar{a}_i$, the following can be obtained:

$$a_i = \frac{a_i^+ + a_i^-}{2}, \qquad \Delta \bar{a}_i = \frac{a_i^+ - a_i^-}{2},$$
$$|\check{a}_i - a_i| = |\Delta a_i| \leq \Delta \bar{a}_i. \quad (2)$$

By using free parameter γ_i interval coefficient $\check{a}_i$ can be expressed as follows:

$$\check{a}_i = a_i + \Delta a_i = a_i + \gamma_i \cdot \Delta \bar{a}_i, \quad \gamma_i \in [-1; 1]. \quad (3)$$

In Eq. (3), non-negative free parameter $\bar{\gamma}$ which is written by

$$|\gamma_i| \leq \bar{\gamma} \in [0; 1], \quad \forall i \quad (4)$$

can be found.

Based on the expression of Eq. (3) for interval coefficients $\check{a}_i$, Eq. (1) can be rewritten as follows:

$$\check{F}(s) = F(s) + \Delta F(s), \quad (5)$$

where $F(s)$ and $\Delta F(s)$ are the nominal and the uncertain parts of interval polynomial $\tilde{F}(s)$, respectively, and written as

$$F(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n,$$

$$\Delta F(s) = \Delta a_0 s^n + \Delta a_1 s^{n-1} + \cdots + \Delta a_{n-1} s + \Delta a_n,$$

$$= \gamma_0 \Delta \bar{a}_0 s^n + \gamma_1 \Delta \bar{a}_1 s^{n-1} +$$

$$\cdots + \gamma_{n-1} \Delta \bar{a}_{n-1} s + \gamma_n \Delta \bar{a}_n.$$

Moreover, the following inequality can be given:

$$|\Delta F(s)| \leq \bar{\gamma} \cdot \max_{\eta_j} |\Delta \overline{F}(s, \eta_j)|, \qquad (6)$$

where

$$\Delta \overline{F}(s, \eta_j) = \nu_0 \Delta \bar{a}_0 s^n + \nu_1 \Delta \bar{a}_1 s^{n-1} +$$

$$\cdots + \nu_{n-1} \Delta \bar{a}_{n-1} s + \nu_n \Delta \bar{a}_n,$$

$$\eta_j = \{\nu_0, \nu_1, \cdots, \nu_n\}_j, \quad \nu_i \in \{-1; 1\},$$

$$(j = 1, 2, 3, \cdots, 2^{n+1}).$$

When analyzing a given transformation (i.e., the mapping of circular contour $\partial \Gamma$ as is shown in Fig. 1) in an s-plane, i.e.,

$$s = \rho e^{j\theta} + \sigma_0 + j\omega_0, \qquad (7)$$

the following chracteristic polynomial with complex coefficients can be obtained:

$$\Phi(j\alpha) = P(\alpha) + jQ(\alpha), \qquad (8)$$

$$P(\alpha) = \tilde{a}_{0,0}\alpha^n + \tilde{a}_{0,1}\alpha^{n-1} + \cdots + \tilde{a}_{0,n-1}\alpha + \tilde{a}_{0,n},$$

$$Q(\alpha) = \tilde{b}_{0,0}\alpha^n + \tilde{b}_{0,1}\alpha^{n-1} + \cdots + \tilde{b}_{0,n-1}\alpha + \tilde{b}_{0,n},$$

$$\tilde{a}_{0,i} \in [a_{0,i}^-; a_{0,i}^+], \quad \tilde{b}_{0,i} \in [b_{0,i}^-; b_{0,i}^+], \quad (i = 0, 1, 2, \cdots, n).$$

Here, α corresponds to the following variable:

$$\alpha = \tan(\theta/2). \qquad (9)$$

3. Discrimination of the Number of Roots

If the μ pieces of the characteristic roots are present in specified circle $\partial \Gamma$, the argument change in the mapping of characteristic polynomial (1) should be $2\mu\pi$, regardless of its interval coefficients.

Assume that $f_0(\alpha) = P(\alpha)$, $f_1(\alpha) = Q(\alpha)$. As for $P(\alpha)/Q(\alpha)$ (or $-Q(\alpha)/P(\alpha)$), the following division algorithm can be used [4]:

$$f_0(\alpha) = f_1(\alpha)q_1(\alpha) - f_2(\alpha),$$

$$f_1(\alpha) = f_2(\alpha)q_2(\alpha) - f_3(\alpha), \qquad (10)$$

$$\cdots$$

$$f_{2n-2} = f_{2n-1}(\alpha)q_{2n-1}(\alpha) - f_{2n}(\alpha).$$

If $f_0(\alpha)$ and $f_1(\alpha)$ are of the n-th order in regard to α and coprime,

$$f_2(\alpha), f_3(\alpha), \cdots, f_{2n}$$

are all present and

$$f_{2h}(\alpha), \quad f_{2h+1}(\alpha), \quad (h = 0, 1, \cdots, n-1)$$

become the $(n-h)$-th order polynomial, and f_{2n} becomes a non-zero constant. That is, $f_2(\alpha), f_3(\alpha), \cdots, f_{2n}$ can be expressed as follows:

$$f_2(\alpha) = a_{1,1}\alpha^{n-1} + \cdots + a_{1,n}$$

$$f_3(\alpha) = b_{1,1}\alpha^{n-1} + \cdots + b_{1,n}$$

$$\cdots \qquad (11)$$

$$f_{2n-2}(\alpha) = a_{n-1,n-1}\alpha + a_{n-1,n}$$

$$f_{2n-1}(\alpha) = b_{n-1,n-1}\alpha + b_{n-1,n}$$

$$f_{2n} = a_{n,n}.$$

In order to simplify the notation, interval coefficients $\tilde{a}_{i,j}$, $\tilde{b}_{i,j}$ (which are obtained by the division algorithm) simply denote $a_{i,j}$, $b_{i,j}$.

Each coefficient can be given by the following sequential operations:

$$a_{1,p} = b_{0,p}\left(\frac{a_{0,0}}{b_{0,0}}\right) - a_{0,p},$$

$$b_{1,p} = a_{1,p+1}\left(\frac{b_{0,0}}{a_{1,1}}\right) - b_{0,p},$$

$$(p = 1, 2, \cdots, n) \qquad (12)$$

$$\cdots$$

$$a_{q,p} = b_{q-1,p}\left(\frac{a_{q-1,q-1}}{b_{q-1,q-1}}\right) - a_{q-1,p},$$

$$b_{q,p} = a_{q,p+1}\left(\frac{b_{q-1,q-1}}{a_{q,q}}\right) - b_{q-1,p},$$

$$(p = q, \cdots, n)$$

$$\cdots$$

$$a_{n,n} = b_{n-1,n}\left(\frac{a_{n-1,n-1}}{b_{n-1,n-1}}\right) - a_{n-1,n},$$

$$(a_{q,n+1} = 0).$$

When a characteristic equation is expressed as an interval polynomial, these sequential operations should be based on interval arithmetic. Interval arithmetic, however, can only be used where each variable (coefficient) is independent. Because the calculation in Eq. (12) was advanced sequentially by using the preceding results, each variable (coefficient) is not independent; therefore, the use of interval arithmetic in the above sequential operation may give a more conservative result. The number of characteristic roots, however, can be discriminated by the extreme point in each coefficient of Eq. (12). (Note that the denominator of Eq. (12) must not become zero, although

this is only a problem in certain singular points.)

Since the extreme points of the interval sets of series (array) $a_{q,p}$ and $b_{q,p}$ ($q = 1, 2, \cdots, n, p = q, \cdots, n$) are given in order according to the extreme points of the uncertain coefficients in interval polynomial (1) and (8), these extreme points can be determined by using all the combinations of the extreme points of the coefficients in the interval polynomial. Therefore, the following theorem in regard to the robustness of control systems can be obtained [3]:

[Theorem] A necessary condition for the μ pieces of the roots of characteristic equation $\tilde{F}(s) = 0$ (which are present in specified circle $\partial\Gamma$) is shown below.

The following coefficient ratios should be calculated (they were present in division algorithm (12)):

$$\frac{b_{0,0}}{a_{1,1}}, \quad \frac{b_{1,1}}{a_{2,2}}, \quad \cdots, \quad \frac{b_{n-1,n-1}}{a_{n,n}}. \tag{13}$$

for all the combinations of the extreme points of the uncertain coefficients in the family of characteristic equations,

$$F(s) = [a_0^-; a_0^+]s^n + [a_1^-; a_1^+]s^{n-1} + \cdots + [a_n^-; a_n^+] = 0. \tag{14}$$

If the number of ratios that is to be negative is not changed, a control system that is characterized by Eq. (14) has a robust performance in regard to the invariance of the number of characteristic roots in specified circle $\partial\Gamma$. Moreover, when the above also holds for $\bar{\gamma} \in [0; 1]$ in Eqs. (2), (3) and (4), it becomes the necessary and sufficient condition.

(Proof) The necessity is obvious from Sturm's theorem (i.e., the results of Eqs. (11) and (12)). An outline of the proof, especially for (c), will be given in **Appendix**.

On the other hand, the sufficiency can be proven by using Rouché's theorem. If the number of roots of characteristic equation

$$F(s) + \gamma \cdot \Delta\overline{F}(s, \eta_j) = 0, \quad \forall\gamma \in [0; \bar{\gamma}], \quad \forall\eta_j \tag{15}$$

is invariant inside the specified contour, the following must holds as to $s \in \partial\Gamma$:

$$F(s) \neq -\gamma \cdot \Delta\overline{F}(s, \eta_j), \quad \forall\gamma \in [0; \bar{\gamma}], \quad \forall\eta_j.$$

Thus, the following inequality can be given:

$$|F(s)| > \bar{\gamma} \cdot \max_{\eta_j} |\Delta\overline{F}(s, \eta_j)|, \quad \forall s \in \partial\Gamma. \tag{16}$$

As a result, we can obtain

$$|F(s)| > |\Delta F(s)|, \quad \forall s \in \partial\Gamma \tag{17}$$
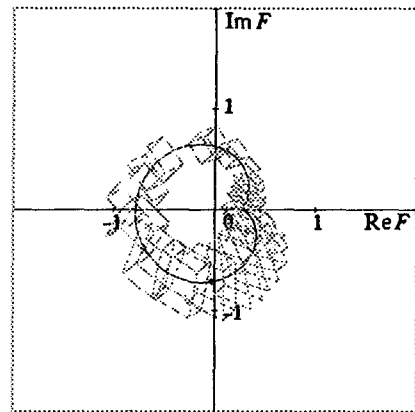
from Eq. (6).

Fig. 2 Polyhedral mapping of $\partial\Gamma$ for Example 4.1.
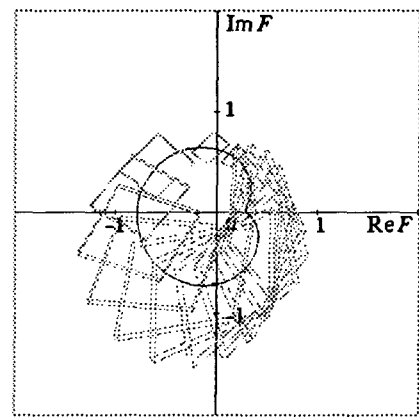

Fig. 3 Polyhedral mapping of $\partial\Gamma$ for Example 4.2.

By using Rouché's theorem we may conclude that the number of roots of characteristic equation

$$\tilde{F}(s) = F(s) + \Delta F(s) = 0 \tag{18}$$

is invariant inside the specified contour, regardless of uncertainty $\Delta F(s)$, i.e., free parameters $\gamma_i \in [-1; 1]$ in Eq. (3). $\square$

[Example 4.1] Consider the family of characteristic equations with low frequency uncertainty (e.g., sector nonlinearities [5, 6]) as follows:

$$0.005s^4 + 0.25s^3 + [0.9; 1.1]s^2 + [1.1; 1.4]s + [0.8; 1.2] = 0. \tag{19}$$

When a circle with center $(-0.75, j)$ and radius $r = 0.6$ is specified, the number of the characteristic roots in the circle is $\mu = 1$ for all $2^3 = 8$ combinations in the extreme points of the uncertain coefficients. A graphical interpretation of the discrimination theorem, i.e., a polyhedral mapping is shown in Fig. 2. In these figures, 2^3 vertices of the polyhedra indicate extreme points of the interval sets [7].
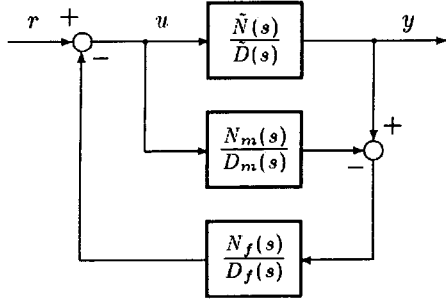
Fig. 4 Realization of robust performace via model feedback.

The calculated results show that the number of roots in the specified circle did not change. Moreover, the number of the dominant roots was maintained for all $\overline{\gamma} \in [0; 1]$. On the other hand, when a circle with radius $r = 0.3$ is specified, the calculated results show that there are some cases where the dominant root does not exist in the specified area.

[Example 4.2]　Consider the family of characteristic equations with high frequency uncertainties as:

$$[0.001; 0.009]s^4 + [0.1; 0.4]s^3 + [0.8; 1.2]s^2 + 1.25s + 1.0 = 0.$$
(20)

When a circle with center $(-0.75, j)$ and radius $r = 0.6$ is specified, the number of the characteristic roots did not change. A polyhedral mapping is as shown in Fig. 2. On the other hand, when a circle with radius $r = 0.3$ is specified, the calculated results show that there are some cases where the dominant root does not exist in the specified area.

4.　Realization of Robust Performance

Let us consider a model feedback system, as is shown in Fig. 4. Here, $\tilde{G} = \dfrac{\tilde{N}(s)}{\tilde{D}(s)}$, $G_m = \dfrac{N_m(s)}{D_m(s)}$, $G_f = \dfrac{N_f(s)}{D_f(s)}$, are uncertain(interval) plants, plant model and feedback compensator, respectively. Here, we will assume that $\tilde{N}(s)$ and $\tilde{D}(s)$ are expressed by interval polynomials as shown in Eq. (1).

By using this type of feedback structure, a robust performance for the uncertain control system can be realized [8]. If feedback compensator G_f is chosen as

$$N_f(s) = D_m(s), \qquad D_f(s) = N_0(s, \tau)N_m(s),$$
(21)

modified control signal u is given by

$$\hat{u}(s) = \frac{N_0(s, \tau)}{N_0(s, \tau) - 1}\left(\hat{r}(s) - \frac{D_m(s)}{N_0(s, \tau)N_m(s)}\hat{y}(s)\right),$$
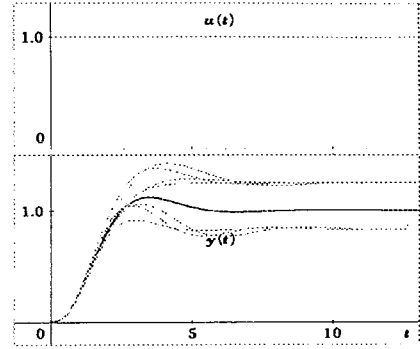(22)

Fig. 5 Step responses for Example 5.1, when model reference feedback was not used.
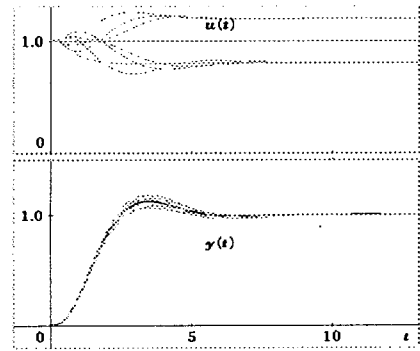

Fig. 6 Step responses for Example 5.1, when model reference feedback was used.

where $\hat{u}(s)$, $\hat{r}(s)$ and $\hat{y}(s)$ denote the Laplace transform of u, r and y, respectively. Here, $N_0(s, \tau)$ is an $(n-m)$-th order polynomial which should be designed in the feedback system as is shown in Fig. 4.

In this paper, we will choose

$$N_0(s, \tau) = (\tau s + 1)^{n-m},$$
(23)

where τ is an appropriate (small) positive number. From Eq. (22) the following can be obtained:

$$(N_0(s, \tau) - 1)\hat{u}(s) = (N_0(s, \tau)\hat{r}(s) - \frac{D_m(s)}{N_m(s)}\hat{y}(s).$$
(24)

As is obvious from Eq. (24), when $\tau s \rightarrow 0$, $\hat{y}(s) \rightarrow \dfrac{N_m(s)}{D_m(s)}\hat{r}(s)$. When operating in lower frequencies, the transfer characteristics from r to y becomes approximately $\dfrac{N_m(s)}{D_m(s)}$ and it is invariant regardless of uncertainties (i.e., interval set parameters) in plant $\tilde{G}$. In other words, it can be shown that the model reference feedback system has a robust performance.

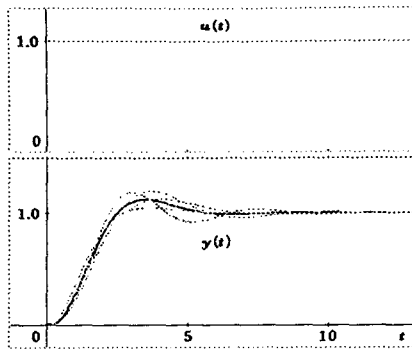The characteristic equation of the closed loop system can

170

Fig. 7 Step responses for Example 5.2, when model reference feedbak was not used.

Fig. 8 Step responses for Example 5.2, when model reference feedbak was used.

be expressed as

$$(N_0(s,\tau) - 1) + \frac{D_m(s)\tilde{N}(s)}{N_m(s)\tilde{D}(s)} = 0. \qquad (25)$$

When uncertainties of plant $\tilde{G}$ exist only in the denominator, that is, $\tilde{N}(s) = N_m(s)$, the characteristic interval polynomial which corresponds to Eq. (1) can be written as follows:

$$\tilde{F}(s) = N_0(s,\tau)D_m(s) + (N_0(s,\tau) - 1)\Delta D(s), \qquad (26)$$

where $\tilde{D}(s) = D_m(s) + \Delta D(s)$. As for Eq. (26) the invariance of the dominant roots is examined in the following examples.

[Example 5.1] Assume that denominator polynomial $\tilde{D}(s)$ of an uncertain plant is expressed by an interval polynomial as shown in Eq. (19) and numerator polynomial $\tilde{N}(s)$ is written by $\tilde{N}(s) = N_m(s) = 1$. In addition, we choose

$$N_0(s,\tau) = (0.1s + 1)^4. \qquad (27)$$

By using this type of model reference feedback, the invariance of the number of characteristic roots can be accomplished in regard to a smaller circle with radius $\rho = 0.3$. Step responses in regard to the extreme points for uncertain control system $\tilde{G}(s)$ are as shown in Fig. 5 and Fig. 6, when the model reference feedback was not used and was used, respectively. As is clear from the figure, the robust performance according to the above was realized.

[Example 5.2] Assume that denominator polynomial $\tilde{D}(s)$ of an uncertain plant is expressed by an interval polynomial as shown in Eq. (20) and numerator polynomial $\tilde{N}(s)$ is written by $\tilde{N}(s) = N_m(s) = 1$. Here, we choose the same $N_0(s,\tau)$ as in Eq. (27). The invariance of the number of characteristic roots can be accomplished in regard to a smaller circle with radius $\rho = 0.3$. Step responses in regard to the extreme points for uncertain control system $\tilde{G}(s)$ are as shown in Fig. 7 and Fig. 8, when the model reference feedback was not used and was used, respectively.
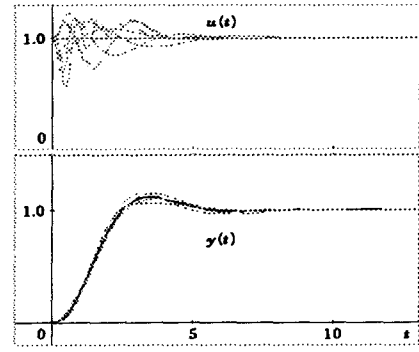
5. Conclusion

This paper discussed the existing area of characteristic roots for control systems which are expressed by transfer functions that are composed of interval polynomials. A discrimination method of the number of characteristic roots in a specified circle on an s-plane was presented, when a characteristic equation was expressed as an interval polynomial. A theorem was given in reference to the extreme point results which corresponds to the weak-Kharitonov's theorem for interval polynomials. The theorem can be used as an invariant condition of the number of characteristic roots in the specified circle. In particular, in this paper, the invariance of the dominant roots in the circlar area and the realization of control systems with robust performance using model reference feedback were examined.

6. References

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Appendix

The equation of the contour (7) can be written as

$$s = \frac{u + v\alpha}{1 - j\alpha}, \tag{28}$$

where

$$u = \rho + \sigma_0 + j\omega_0,$$
$$v = \omega_0 + j(\rho - \sigma_0).$$

A numerator polynomial for variable α,

$$\Phi(j\alpha) = (1 - j\alpha)^n F(j\alpha), \tag{29}$$

that is, Eq. (8) is obtained by substituting Eq. (28) into Eq. (1). Here, the relationship between variable θ and variable α is expressed as:

$$\theta = -\pi; \quad \alpha = -\infty,$$
$$\theta = 0; \quad \alpha = 0,$$
$$\theta = +\pi; \quad \alpha = +\infty.$$

from Eq. (9). Thus argument change $2\mu\pi$ as mention in section 3 becomes $(2\mu - n)\pi$ for Eqs. (8) and (29) by adding change $-n\pi$ in the argument of $(1 - j\alpha)^n$. When P/Q (or $-Q/P$) is considered, the number of sign changes which cross zero for $\alpha : -\infty \rightarrow +\infty$ is $n - 2\mu$. If the number of sign changes which cross the zero of $f_0(\alpha)/f_1(\alpha)$ for $\alpha : \alpha_1 \rightarrow \alpha_2$ is expressed as $N(\alpha_1, \alpha_2)$ and the number of sign changes of sequence

$$f_0(\alpha), \quad f_1(\alpha), \quad \cdots, \quad f_{2n}$$

is expressed as $V(\alpha)$, the following relationship is obtained:

$$N(\alpha_1, \alpha_2) = V(\alpha_1) - V(\alpha_2). \tag{30}$$

Since the condition is $N(-\infty, +\infty) = n - 2\mu$,

$$V(-\infty) - V(+\infty) = n - 2\mu \tag{31}$$

are obtainable.

On the other hand, the following expressions are derived from Eqs. (8) and (10):

$$\lim_{\alpha \to -\infty} \frac{f_0(\alpha)}{f_1(\alpha)} = \frac{a_{0,0}}{b_{0,0}}$$

$$\lim_{\alpha \to -\infty} \frac{f_1(\alpha)}{|\alpha| f_2(\alpha)} = -\frac{b_{0,0}}{a_{1,1}}$$

$$\cdots$$

$$\cdots$$

$$\lim_{\alpha \to -\infty} \frac{f_{2n-1}(\alpha)}{|\alpha| f_{2n}(\alpha)} = -\frac{b_{n-1,n-1}}{a_{n,n}}$$

$$\lim_{\alpha \to +\infty} \frac{f_0(\alpha)}{f_1(\alpha)} = \frac{a_{0,0}}{b_{0,0}} \tag{32}$$

$$\lim_{\alpha \to +\infty} \frac{f_1(\alpha)}{|\alpha| f_2(\alpha)} = \frac{b_{0,0}}{a_{1,1}}$$

$$\cdots$$

$$\cdots$$

$$\lim_{\alpha \to +\infty} \frac{f_{2n-1}(\alpha)}{|\alpha| f_{2n}(\alpha)} = \frac{b_{n-1,n-1}}{a_{n,n}}$$

The condition of Eq. (31) corresponds to observing whether the even numbers of these ratios (ratios to the polynomial of different orders) are negative or not. Consider the following ratio sequence:

$$\lim_{\alpha \to +\infty} \frac{f_1(\alpha)}{|\alpha| f_2(\alpha)}, \quad \cdots, \quad \lim_{\alpha \to +\infty} \frac{f_{2n-1}(\alpha)}{|\alpha| f_{2n}(\alpha)}.$$

In this sequence, if we assume that the number of negative ratios is N and the number of positive ratios is P, the following expression can be obtained from Eq. (31).

$$P - N = n - 2\mu. \tag{33}$$

Since $P + N = n$, $N = \mu$ can be obtained. The coefficients in Eq. (32) can be calculated by

$$\Phi(j\alpha) = \sum_{k=0}^{n} a_k (u + v\alpha)^{n-k} (1 - j\alpha)^k \tag{34}$$

from Eq. (28). This equation is expanded by

$$\sum_{m=0}^{n} \sum_{k=0}^{n} \sum_{l=k-m}^{n-m} a_k \binom{n-k}{n-m-l} \binom{k}{l} (-j)^l u^{m-k+l} v^{n-m-l} \alpha^{n-m}.$$

Therefore, we obtain

$$a_{0,m} + j b_{0,m}$$

$$= \sum_{k=0}^{n} \sum_{l=k-m}^{n-m} a_k \binom{n-k}{n-m-l} \binom{k}{l} (-j)^l u^{m-k+l} v^{n-m-l}$$

where small

$$\binom{k}{l} = {}_k C_l$$

denotes the combination sign.