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Design of LMI-Based H_∞ Controller for Robot Manipulators

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Abstract

In this paper, we present new control method for robot manipulators. The design objective can be the implementation of minimax controller with H_∞ performance via LMI approach to guarantee the robustness and to obtain the exact tracking performance for robot manipulators with system parameter uncertainty and exogenous disturbance.

We show that the Algebraic Riccati equation (ARE) which is needed for the construction of H_∞ controller can be recast into the Algebraic Riccati Inequality (ARI) and the optimal control gain can be obtained by convex optimization method. Then, we will apply the proposed controller to rigid robot manipulators for verifying the performance of our controller.

1. Introduction

Since the last several decades, a great deal of researches have paid attention to the motion control of robot manipulators[1]. There are a lot of control approaches using the conventional control theory for the linearized model of rigid robot system[2]. Especially, Johansson showed that the tracking problem of robot manipulator can be converted to the quadratic optimization problem and the optimal control input value can be obtained from the solution of algebraic Riccati equation which is derived from Hamilton-Jacobi equation[3]. To guarantee the robust control performance for robot systems with the perturbation of linearized system model and exogenous disturbance, Chen proposed state-feedback H_∞ control which shows the stable control performance for the combined disturbances of system[3].

However, the cost function used in the conventional controller did not consider the overall system dynamics appropriately and the solution of algebriac Riccati equation have the limitation that can not deal with the dynamic features of system posed by various disturbances. From this motivation, we investigate the new derivation of the solution of the nonlinear Riccati equation based on the new cost function which take the overall features of system in order to obtain the better control performance.

To derive the solution of Riccati equation efficiently, we take the convex optimization method i.e. Linear Matrix Inequalities(LMIs) which recently have received considerable attention for control analysis and design[4]. A large number of control design problems can be formulated as LMIs.[5] Especially, the ARE for H_∞ control problem can be converted into an ARI and into a LMI due to the bounded real lemma. Such problems can be solved in a time that is comparable to the time required to solve the same ARE[6].

In this paper, we show that the quadratic optimization problem for robot manipulator can be recast as convex optimization problems that involve LMIs. To solve the proposed LMIs problem, we introduce the scaling parameter and recast the proposed LMI problem as the generalized eigenvalue problem (GEVP). By these process, we can design the LMI-based H_∞ controller for robot manipulator which has the robustness and exact tracking performance for parameter uncertainty and exogenous disturbance. Also we apply our proposed controller to a 2-link robot manipulator. Through computer simulation, we will verify the performance of the proposed controller for robot manipulators.

2. H_∞ Model Reference Control and Problem Formulation

2.1. Model Description for Robot Manipulator

The dynamic equation for N-link robot manipulators is given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{1}$$

where q is the vector of N-link coordinates, $\tau \in R^n$ is the vector of the externally applied torque along the directions of their corresponding generalized coordinates q, $M(q) \in R^{n\times n}$ is the positive definite symmetric inertia matrix, $C(q,\dot{q})\dot{q}$ is the vector grouping the Coriolis, centrifugal forces which is the skew symmetric and $G(q)$ is the gravitational forces.

Since the practical robot systems contain the parameter uncertainty of the plant model and the external disturbance inevitably, it is important to consider the effect of the system performance due to the plant uncertainty and the external disturbance. So, we should consider the dynamic equation taking into these effects for real robot systems as the following equation

$$\begin{aligned}&[M_0(q) + \Delta M(q)]\ddot{q} + [C_0(q,\dot{q}) + \Delta C(q,\dot{q})]\dot{q} \\ &+ [G_0(q) + \Delta G(q)] = \tau + w\end{aligned} \tag{2}$$

where w is the finite energy exogenous disturbance, $M_0(q)$, $C_0(q,\dot{q})$, $G_0(q)$ are the nominal system model, ΔM is the uncertainty of the $M(q)$ due to the changes of load, ΔC and ΔG are the perturbations of $C_0(q,\dot{q})\dot{q}$ and $G_0(q)$ due to the changes of the total load.

Therefore, The total uncertain disturbance of system can be combined as

$$\delta = -(\Delta M(q)\ddot{q} + \Delta C(q,\dot{q})\dot{q} + \Delta G(q) - w) \tag{3}$$

By using of Eqn. (3), Eqn. (2) can be showed as the following nominal dynamic equation.

$$M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + G_0(q) = \tau + \delta \tag{4}$$

It is assumed that the desired reference trajectory for system is generated from the following reference model

$$\ddot{q}_r + K_v\dot{q}_r + K_p q_r = K_r r \tag{5}$$

where q_r is a twice continuous differentiable reference trajectory position, $\dot{q}_r$ and $\ddot{q}_r$ are the corresponding velocity and acceleration with some bounded driving signal r.

The state tracking error is defined as

$$\tilde{x} = \begin{bmatrix}\dot{\tilde{q}} \\ \tilde{q}\end{bmatrix} = \begin{bmatrix}\dot{q} - \dot{q}_r \\ q - q_r\end{bmatrix} \tag{6}$$

Then, The tracking problem of robot manipulator with uncertain disturbance can be reduced to the regulation problem of error state $\tilde{x}$. Using Eqn. (5) and (6), the state equation for the error state $\tilde{x}$ is obtained as the following form

$$\dot{\tilde{x}} = A_T(\tilde{x},t)\tilde{x} + B_T(\tilde{x},t)u + B_T(\tilde{x},t)d \tag{7}$$

where

$$A_T := T_0^{-1}\begin{bmatrix} -M_0(q)C_0(q,\dot{q}) & 0_{n\times n} \\ T_{11}^{-1} & -T_{11}^{-1}T_{12}\end{bmatrix}T_0$$

$$B_T(\tilde{x},t) := T_0^{-1}\begin{bmatrix} I_{n\times n} \\ 0_{n\times n}\end{bmatrix}M_0^{-1}(q)$$

$$d := M_0(q)T_{11}M_0^{-1}(q)\delta$$

Then, control input u is given as

$$u := \begin{bmatrix} M(q) & C(q,\dot{q})\end{bmatrix}\begin{bmatrix}\dot{\tilde{z}}_1 \\ \tilde{z}_1\end{bmatrix} = M(q)T_1\dot{\tilde{x}} + C(q,\dot{q})T_1\tilde{x} \tag{8}$$

The matrix T_1 is introduced via the following state-space transformation of $\tilde{x}$

$$\tilde{z} = \begin{bmatrix}\tilde{z}_1 \\ \tilde{z}_2\end{bmatrix} = T_0\tilde{x} = \begin{bmatrix}T_1 \\ T_2\end{bmatrix}\begin{bmatrix}\dot{\tilde{q}} \\ \tilde{q}\end{bmatrix} = \begin{bmatrix}T_{11} & T_{12} \\ 0_{n\times n} & I_{n\times n}\end{bmatrix}\begin{bmatrix}\dot{\tilde{q}} \\ \tilde{q}\end{bmatrix} \tag{9}$$

Since the disturbance of system in Eqn. (7) is uncertain, we can take the torque for robot manipulator with just nominal system parameter. From Eqn. (4) and (5), we get the applied torque τ as the following equation.

$$\begin{aligned}\tau = {} & M_0(q)(\ddot{q}_r - T_{11}^{-1}T_{12}\dot{\tilde{q}} - T_{11}^{-1}M_0(C(q,\dot{q})B^T T_0\tilde{x} - u)) \\ & + C_0(q,\dot{q})\dot{q} + G_0(q)\end{aligned} \tag{10}$$

where

$$B = \begin{bmatrix} I_{n\times n} & 0_{n\times n}\end{bmatrix}^1$$

2.2. Problem Formulation with H_∞ Performance

In order to cancel the effect of disturbance for robot manipulator, The H_∞ control problem use the following performance criterion including a desired disturbance attenuation level γ for the tracking error dynamic equation as

$$V(\tilde{x}(t),t)=\min_{u(\cdot)\in L_2}\max_{0\neq d(\cdot)\in L_2}\frac{\int_0^\infty(\frac{1}{2}\tilde{x}^T(t)Q\tilde{x}(t)+\frac{1}{2}u^T(t)Ru(t))dt}{\int_0^\infty(\frac{1}{2}d^T(t)d(t))dt}\leq\gamma^2 \quad (11)$$

which has the initial condition $\tilde{x}(0)=0$.

Eqn. (11) is equivalent to the H_∞ minimax problem[7]. To solve the minimax problem, we define the following cost functional as

$$J(\tilde{x}(t),u,dt)=:\int_t^\infty L(\tilde{x}(s),u(s),d(s))ds \quad (12)$$

$$L(\tilde{x},u,d)=\frac{1}{2}\tilde{x}^T(t)Q\tilde{x}(t)+\frac{1}{2}u^T(t)Ru(t)-\frac{1}{2}\gamma^2 d^T(t)d(t)$$

By introducing the value function $V(\tilde{x}(t),t)=\min\max J(\tilde{x}(t),u,d,t)$, the performance criterion of Eqn. (11) is equivalent to

$$V(\tilde{x}(0),0)=\min\max J(\tilde{x}(0),u,d,0)\leq 0\ ,\ \tilde{x}(0)=0 \quad (13)$$

where the $V(\tilde{x},t)$ must satisfy the following terminal constraint

$$V(\tilde{x}(\infty),\infty)=0 \quad (14)$$

Therefore, the performance criterion (13) is achieved by the optimal control input u^* and the worst case disturbance d^*. The solution of the performance criterion of (11) needs the following minimax Bellman-Isaacs Eqn. [8] as

$$-\frac{\partial V(\tilde{x},t)}{\partial t}=\min\max\left\{L(\tilde{x},u,d)+\left(\frac{\partial V(\tilde{x},t)}{\partial\tilde{x}}\right)^T\dot{\tilde{x}}\right\} \quad (15)$$
$$:=\min\max H(\tilde{x},u,d,t):=H^*(\tilde{x},u^*,d^*,t)$$

where H is the Hamiltonian.

The value function $V(\tilde{x},t)$ is chosen as

$$V(\tilde{x},t):=\frac{1}{2}\tilde{x}^T P(\tilde{x},t)\tilde{x} \quad (16)$$

where

$$P(\tilde{x},t)=T_0^T S T_0$$

and S is the positive definite diagonal matrix.

After some algebraic computation, we can obtain the following algebriac Riccati equation from Eqn. (15).

$$\dot{P}(\tilde{x},t)+P(\tilde{x},t)A_T(\tilde{x},t)+A_T^T(\tilde{x},t)P(\tilde{x},t)$$
$$-P(\tilde{x},t)B_T(\tilde{x},t)\left(R^{-1}-\frac{1}{\gamma^2}I\right)B_T^T(\tilde{x},t)P(\tilde{x},t)+Q=0 \quad (17)$$

where $(R^{-1}-\frac{1}{\gamma^2}I)$ is to be positive.

Finally, we provide the main result of this section.

Theorem 1: Subject to the perturbed error dynamics (7), let it be given nonsingular T_0. The H_∞ control problem (11) is solvable if $\gamma^2 I>R$ and the algebraic Riccati equation (17) has a positive definite symmetric matrix $P(\tilde{x},t)$. The optimal control u^* and the worst case disturbance d^* are given as

$$u^*=-R^{-1}B_T^T P(\tilde{x},t)\tilde{x}\ ,\ w^*=\frac{1}{\gamma^2}B_T^T P(\tilde{x},t)\tilde{x} \quad (18)$$

Also, from Eqn. (10), the corresponding applied torque which guarantees the desired H_∞ performance is given by

$$\tau^*=M_0(q)(\ddot{q}_r-T_{11}^{-1}T_{12}\dot{\tilde{q}}-T_{11}^{-1}M_0(C(q,\dot{q})B_T{}^T P\tilde{x}-u^*)) \quad (19)$$
$$+C_0(q,\dot{q})\dot{q}+G_0(q)$$

Proof: See the reference [3]

3. LMI–Based H_∞ Controller

In this section, we show that the algebrai Riccati equation (19) can be recast as generic LMIs problem through simple variable change and the optimal Riccati gain can be obtained by the convex optimization method.[8]

By using Eqn. (16), Eqn. (17) can be rewritten as

$$\dot{S}(\tilde{x},t)+S(\tilde{x},t)A(\tilde{x},t)+A^T(\tilde{x},t)S(\tilde{x},t)$$
$$-S(\tilde{x},t)B(\tilde{x},t)\tilde{R}^{-1}B^T(\tilde{x},t)S(\tilde{x},t)+\tilde{Q}=0 \quad (20)$$

where $\tilde{R}^{-1}=(R^{-1}-\frac{1}{\gamma^2}I)$, $\tilde{Q}=T_0^{T^{-1}}QT_0^{-1}$

For given disturbance attenuation level γ, the control problem can be solved if we can find $S(\tilde{x},t)$ that satisfy the following inequality ,

$$\begin{aligned} &S(\tilde{x},t)A(\tilde{x},t)+A^{\mathrm{T}}(\tilde{x},t)S(\tilde{x},t)\\ &-S(\tilde{x},t)B(\tilde{x},t)\tilde{R}^{-1}B^{\mathrm{T}}(\tilde{x},t)S(\tilde{x},t)+\tilde{Q}<0 \end{aligned} \tag{21}$$

However, Eqn. (21) can not be recast the generic LMIs problem since $\tilde{R}$ must be positive definite matrix.

To satisfy the LMI constraint condition, we can change variable $S(\tilde{x},t)$ to $X^{-1}(\tilde{x},t)$. Through algebraic process, we can derive the following inequality from the Eqn. (21).

$$\begin{aligned} &A(\tilde{x},t)X(\tilde{x},t)+X(\tilde{x},t)A^{\mathrm{T}}(\tilde{x},t)\\ &-B(\tilde{x},t)\tilde{R}^{-1}B^{\mathrm{T}}(\tilde{x},t)+X(\tilde{x},t)\tilde{Q}X(\tilde{x},t)<0 \end{aligned} \tag{22}$$

which is the generic LMIs and is equivalent to the following equation.

$$\begin{bmatrix} A(\tilde{x},t)X(\tilde{x},t)+X(\tilde{x},t)A^{\mathrm{T}}(\tilde{x},t)-B(\tilde{x},t)\tilde{R}^{-1}B^{\mathrm{T}}(\tilde{x},t) & X(\tilde{x},t)\\ X(\tilde{x},t) & -\tilde{Q}^{-1} \end{bmatrix}<0 \tag{23}$$

To solve Eqn (23), we introduce new auxiliary variable Z satisfying $S(\tilde{x},t)<Z(\tilde{x},t)$ and $\mathrm{Tr}(Z)<\lambda$. With the variable Z, we obtain the following analysis result: A is stable and $\min\max J(\tilde{x}(0),u,d,0)\le 0$ iff there exist symmetric $X:=S^{-1}$ and Z such that

$$\begin{gathered} \begin{bmatrix} X(\tilde{x},t)A(\tilde{x},t)+A^{\mathrm{T}}(\tilde{x},t)X(\tilde{x},t)-B(\tilde{x},t)\tilde{R}^{-1}B^{\mathrm{T}}(\tilde{x},t) & A(\tilde{x},t)\\ A(\tilde{x},t) & -\tilde{Q}^{-1} \end{bmatrix}<0\\ \begin{bmatrix} Z & I\\ I & X \end{bmatrix}>0\\ trace(Z)<\lambda \end{gathered} \tag{24}$$

Then, we can solve the new LMIP by generalized eigenvalue minimization among the generic LMI problems. The new LMI constraint can be summarized as

$$\text{LMIP : minimize } \lambda \text{ subject to } \Gamma(\lambda,X,Z)>0$$
$$\Gamma(\lambda,X,Z)=diag(\Gamma_1\;\Gamma_2\;\Gamma_3)$$

and the block diagonal matrices are

$$\begin{aligned} &\Gamma_1(\lambda,X,Z)\\ &=\begin{bmatrix} -X(\tilde{x},t)A(\tilde{x},t)-A^{\mathrm{T}}(\tilde{x},t)X(\tilde{x},t)+B(\tilde{x},t)\tilde{R}^{-1}B^{\mathrm{T}}(\tilde{x},t) & A(\tilde{x},t)\\ A(\tilde{x},t) & \tilde{Q}^{-1} \end{bmatrix}\\ &\Gamma_2(\lambda,X,Z)=\begin{bmatrix} Z & I\\ I & X \end{bmatrix}>0\\ &\Gamma_3(\lambda,X,Z)=-trace(Z)+\lambda \end{aligned} \tag{25}$$

4. Computer Simulation

For computer simulation, we consider the 2-link robot manipulator as shown in figure 1,

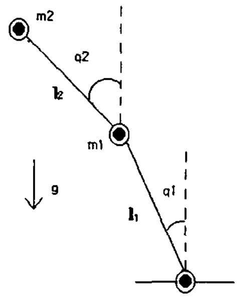

Fig. 1. The 2-link robot manipulator

The parameters of dynamic equation for 2-link robot manipulator is given as

$$\begin{aligned} &M_0(q)=\begin{bmatrix} (m_1+m_2)\,l_1^2 & m_2 l_1 l_2(s_1 s_2+c_1 c_2)\\ m_2 l_1 l_2(s_1 s_2+c_1 c_2) & m_2 l_2^2 \end{bmatrix}\\ &G_0(q)=m_2 l_1 l_2(c_1 s_2-s_1 c_2)\begin{bmatrix} 0 & -\dot{q}_2\\ -\dot{q}_1 & 0 \end{bmatrix}\\ &G_0(q)=\begin{bmatrix} -(m_1+m_2)\,l_1\,g s_1\\ -m_2 l_1\,g s_s \end{bmatrix} \end{aligned} \tag{26}$$

where $c_1=cos(q_1)$, $s_1=sin(q_1)$, $m_1=1$, $m_2=10$, $l_1=l_2=1$, and $\dot{q}_1(0)=0$, $\dot{q}_2(0)=0$ $q_1(0)=-1$, $q_2(0)=-1$.

The parameters of the desired reference trajectory in Eqn. (5) are given by

$$q_r(0)=[0\;\;0]^{\mathrm{T}},\quad K_v=0_{n\times n},\quad K_p=K_r=I_{n\times n} \tag{27}$$

Suppose that the plant perturbation and exogenous disturbance are given by

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 5\ddot{q}_1 + 5(s_1 s_2 + c_1 c_2)\ddot{q}_2 - 5(c_1 s_2 - s_1 c_2)\dot{q}_1^2 \dot{q}_2 - 5g s_2 + w_1 \\ 5\ddot{q}_2 + 5(s_1 s_2 + c_1 c_2)\ddot{q}_1 - 5(c_1 s_2 - s_1 c_2)\dot{q}_2^2 \dot{q}_1 - 5g s_2 + w_2 \end{bmatrix} \quad (28)$$

When H_∞ controller is constructed by the proposed LMIP under the condition of desired disturbance attenuation level $\gamma = 0.2$ and $\gamma = 1$, the transformation matrix T_0 is given by

$$T_0 = \begin{bmatrix} 0.5 & 0 & 5 & 0 \\ 0 & 0.5 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The convex optimization computation is performed with the function gevp from the LMI Control Toolbox. Through the computation, the optimal Riccati control gains are obtained as

$$\text{Case 1}: \quad S = \begin{bmatrix} 3.5118 & 2.7239 & 0 & 0 \\ 2.7239 & 3.2273 & 0 & 0 \\ 0 & 0 & 5.7950 & 0.3183 \\ 0 & 0 & 0.3183 & 5.7618 \end{bmatrix}$$

$$\text{Case 2}: \quad P = \begin{bmatrix} 3.3434 & 2.6318 & 0 & 0 \\ 2.6318 & 2.0686 & 0 & 0 \\ 0 & 0 & 5.6190 & 0.2343 \\ 0 & 0 & 0.2343 & 5.5946 \end{bmatrix}$$

Figure 2 and 3 show the control performance of the angular positions $q_1(t)$ and the angular velocity $\dot{q}_1(t)$ for case 1 and 2. It is shown that the proposed control method can control robot manipulator efficiently with various disturbance levels. From the simulation results, it is shown that the proposed LMI-based controller can take the optimal Riccati solution by convex optimization method and the controller can achieve the exact tracking performance to the desired disturbance attenuation level for robot manipulator.

5. Conclusion

In this paper, we propose the LMI-based H_∞ controller which have the exact tracking performance for robot manipulator with system parameter uncertainty and exogenous disturbance. It is shown that the nonlinear Riccati equation can be recast as the LMIP constraints and optimal solution of Riccati equation can be determined by the proposed LMIP. Since LMI constraints deal with the overall features, the explicit solution can be obtained. Through computer simulation, we can verify that our proposed controller shows more exact tracking performance than the conventional robotic H_∞ controller for 2-link rigid robot manipulator and also satisfies the desired disturbance attenuation levels.

6. References

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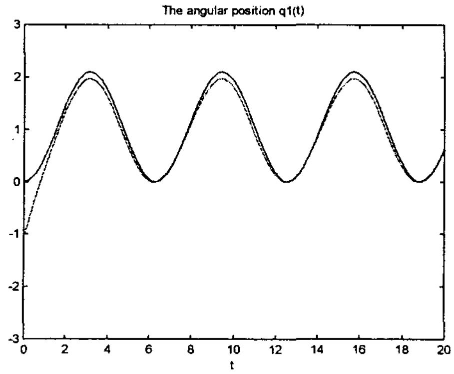
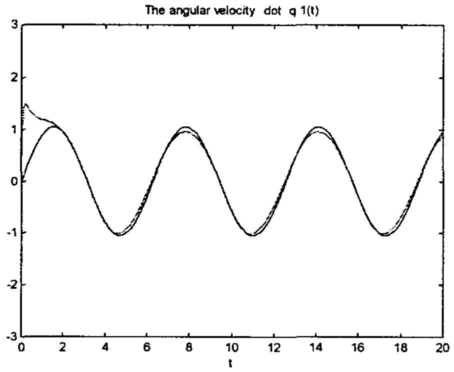
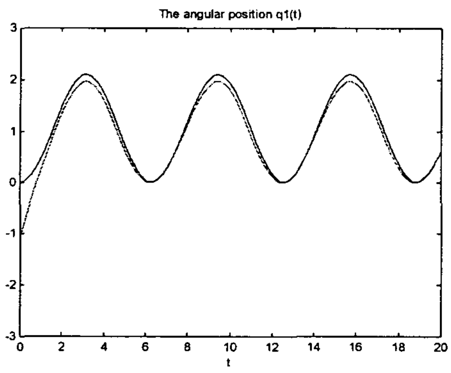


(a) $q_1(t)$

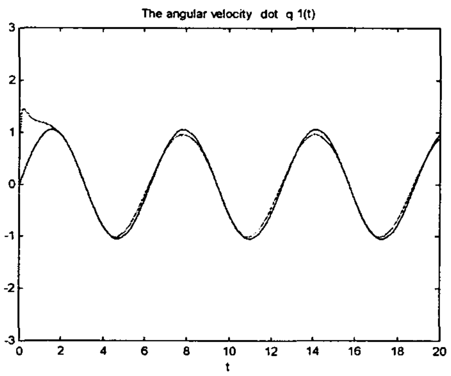


(b) $\dot{q}_1(t)$

Fig. 2. The angular position of $q_1(t)$ and velocity $\dot{q}_1(t)$ with disturbance attenuation $\gamma = 0.2$



(a) $q_1(t)$



(b) $\dot{q}_1(t)$

Fig. 3. The angular position of $q_1(t)$ and velocity $\dot{q}_1(t)$ with disturbance attenuation $\gamma = 1$