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A Simple Method for Frequency Domain Identification

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Abstract

In this paper, a simple method is presented to synthesize a transfer function from experimentally obtained gain and phase data. The method we offer here is based on the previous method given by M.Hassul etc. [1], where they proposed relevant formulas in a straightforward manner so that undergraduate students could follow the development more easily. This method, however, inevitably is accompanied by a significant difference between the real and identified model especially in the low frequency region. We solve this problem by introducing a new weighting function that can be determined by using the additive uncertainty of the identified transfer function.

1. Introduction

Most undergraduate students in electrical engineering programs learn to represent systems in the frequency domain. Single input/single output systems are described by the ratio of the Laplace transform of their input and output signals. The resulting complex transfer function is usually plotted as Bode magnitude and phase diagram. In the frequency domain, we can easily vary the frequency of the input signal over a range of interest and analyze the resulting response.

System identification is the reverse process. Given the data in the frequency domain, one can estimate the system transfer function. Generally, students use straight-line approximation to determine the break points of the Bode magnitude curve. With this knowledge and low frequency gain, the transfer function can be roughly reconstructed.

System Identification including time or frequency domain is, of course, sophisticated. There are plenty of good system identification tools or algorithms by which we can obtain a better model for the control system design, it seems, however, that they generally require quite high-level mathematical knowledge and even more an increase in the input-output data is unavoidable in improving the quality of the obtained model. Therefore, these tools are not likely to be adequate to identify a simple plant, which is used primarily for the undergraduate course.

The method we offer here is basically using the previous method given by M.Hassul etc. [1], where they proposed relevant formulas in a straightforward manner so that undergraduate students could more easily follow the development. Their method, however, inevitably is accompanied by a significant difference between the real and identified model especially in the low frequency region. We overcame this difficulty by introducing a new weighting function that could be

obtained from the experimental data.

In Section II, Hassul's method for frequency domain identification will be described, by which we can have a general idea of the given plant. In Section III, we will derive the general approach to determine an weighting function for better identification. Then, we will present several examples to prove its effectiveness in Section IV. Finally in Section V, we will discuss the limit to this method and refer to some possibilities for future improvement.

2. Hassul's Method for Identification

In this section, we introduce the fundamental concept on which the Hassul's method is based, and describe its derivation process to compute a transfer function from the data.

For a given set of experimentally determined magnitude and phase data, we assumed that the following transfer function is a reasonable model:

$$(j\omega)^k G(j\omega) = |G(j\omega)|e^{j\vartheta(\omega)} = \frac{A(j\omega)}{B(j\omega)} = \frac{a_m(j\omega)^m + a_{m-1}(j\omega)^{m-1} + \cdots + a_1(j\omega) + 1}{b_n(j\omega)^n + b_{n-1}(j\omega)^{n-1} + \cdots + b_1(j\omega) + b_o} \quad (1)$$

where n>m, and k is the number of poles (k>0) at the origin or the number of zeros (k<0) at the origin. Temporarily we regard k as 0 to simplify the derivation process.

It is important to note that the user of this technique must preselect the order of the numerator and denominator. Although there are several methods [2] to estimate the order of a LTI system, it generally requires high level mathematical background which is beyond the level of most undergraduate students. Therefore, the user can make several runs with different values for m and n, and then decide the best value which produces the greatest accuracy. The goal of identification is to estimate the real coefficients $(a_1, a_2, \cdots, a_m, b_0, b_1, \cdots, b_n)$ of the rational transfer function Eq.(1) using experimentally measured frequency response data (magnitude and phase). The estimate is found by minimizing the error function, as in this example

$$\varepsilon_1 = G(j\omega) - \frac{A(j\omega)}{B(j\omega)} \quad (2)$$

which was used by Levy [3]. The quality of the identified model is principally dependent on how to define the error function. Until now, various types of error function were proposed [4].

Hassul took the error function such as

$$\varepsilon_2 = \frac{1}{G(j\omega)} - \frac{B(j\omega)}{A(j\omega)} \quad (3)$$

which defines the error based on the inverse of Eq.(2). This seems quite useful because it simplifies the mathematics. Eq.(1) can be rewritten as

$$b_0 + b_1(j\omega) + b_2(j\omega)^2 + \cdots + b_n(j\omega)^n = (1 + a_1(j\omega) + a_2(j\omega)^2 + \cdots + a_m(j\omega)^m)(R + jX) \quad (4)$$

where

$$R(\omega) = \frac{\cos(\theta(\omega))}{|G(j\omega)|} \text{ and } X(\omega) = \frac{-\sin(\theta(\omega))}{|G(j\omega)|} \quad (5)$$

Bringing the unknown coefficients to the left side of the equation gives us

$$\sum_{i=0}^{n} b_i(j\omega)^i - (\sum_{k=1}^{m} a_i(j\omega)^k)(R + jX) = R + jX \quad (6)$$

We now gather all of the real terms on the left side of Eq.(6) and equate them to R, and the imaginary terms are equated to X. At this point, to allow for n and m to be either odd or even, some new indexes are introduced.

$$p = [n/2], q = [(n-1)/2], r = [m/2], s = [(m-1)/2]$$

where $[\bullet]$ returns the integer value of the argument. Then, we can represent Eq.(4) as two basic equations with (n+m+1) unknowns by rearranging Eq.(6). It follows:

$$b_0 - \omega_i^2 b_2 + \omega_i^4 b_4 + \cdots + (-1)^p \omega_i^{2p} b_{2p} + a_2\omega_i^2 R(\omega_i) - a_4\omega_i^4 R(\omega_i) + \cdots + (-1)^{2r+1} a_{2r}\omega_i^{2r} R(\omega_i) + a_1\omega_i X(\omega_i) - a_3\omega_i^3 X(\omega_i) + \cdots + a_{2s+1}(-1)^s \omega_i^{2s+1} X(\omega_i) = R(\omega_i) \quad (7)$$

$$b_1\omega_i - b_3\omega_i^3 + b_5\omega_i^5 + \cdots + (-1)^q b_{2q+1}\omega_i^{2q+1} + a_2\omega_i^2 X(\omega_i) - a_4\omega_i^4 X(\omega_i) + \cdots + (-1)^{2r+1} a_{2r}\omega_i^{2r} X(\omega_i) - a_1\omega_i R(\omega_i) + a_3\omega_i^3 R(\omega_i) + \cdots + (-1)^s a_{2s+1}\omega_i^{2s+1} R(\omega_i) = X(\omega_i)$$

(8)

Eq.(7) and (8) are iterated as many times as the number of the measured data. These equations can be represented in a matrix form as follows:

$$\begin{bmatrix} \Omega_1 & 0 & \Omega_2 & \Omega_3 \\ 0 & \Omega_4 & \Omega_5 & \Omega_6 \end{bmatrix} \mathbf{Z} = \begin{bmatrix} \mathbf{R} \\ \mathbf{X} \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} b_0 & \cdots & b_{2p} & b_1 & \cdots & b_{2q+1} & a_2 & \cdots & a_{2r} & a_1 & \cdots & a_{2s+1} \end{bmatrix}^T \quad (9)$$

where the coefficients vectors is partitioned as b_{even}, b_{odd}, a_{even} and a_{odd}. The size of each submatrix is

$$\Omega_1 : g \times (p+1), \Omega_2 : g \times r, \Omega_3 : g \times (s+1)$$

$$\Omega_4 : g \times (q+1), \Omega_5 : g \times s, \Omega_6 : g \times (s+1)$$

where g is the number of measurements. Eq.(9) is simplified as

$$\Omega \mathbf{P} = \mathbf{V} \quad (10)$$

where **P** is the vector of coefficients and **V** is the vector of measurements. Generally, the number of measurements is larger than (n+m+1) because we would want to use more output data to reduce measurement error due to noise. In this case, the matrix Ω in Eq.(10) would now have more rows than columns.

One solution given by Hassul was to use a pseudo-inverse. By making use of this, the coefficients of the numerator and denominator could be obtained as follows:

$$\mathbf{P} = (\Omega^T \Omega)^{-1} \Omega^T \mathbf{V} \quad (11)$$

It is well-known that Eq.(11) provides a least-square fit of the coefficients to the data points. We adopted Eq.(11) as a method to acquire the approximate information on the transfer function of a given plant.

3. Determination of Weighting Functions

In this section, we propose one method to obtain a weighting function, by which the better low-frequency fit can be accomplished.

Hassul's method (Eq.(7) and (8)), as well as other similar techniques, usually experiences significant errors at low frequencies. The problem is that the errors are biased and the elements of the matrix Ω are functions of powers of ω. For low frequencies, these terms can become infinitely small. It is, however, difficult to show mathematically how the error is related to the frequency ω on Eq. (11).

Hassul's solution to reduce the low-frequency gap is to multiply Eq.(7) and (8) by appropriate scale factors. These scale factors are generally large for low frequencies and small for high frequencies. They represented this weighting factors in matrix form, and multiplied this on both sides of Eq.(10). Then it becomes

$$\mathbf{W}\Omega\mathbf{P} = \mathbf{W}\mathbf{V} \quad (12)$$

where **W** is usually a diagonal matrix with elements equal to the scale factors we mentioned above. The weighted pseudo-inverse is

$$\mathbf{P} = (\Omega^T \mathbf{W}^T \mathbf{W} \Omega)^{-1} \Omega^T \mathbf{W}^T \mathbf{W} \mathbf{V} \quad (13)$$

Unfortunately, there are no good methods to decide a suitable weighting function based on the data. Hassul sometimes chose a scale factor (not a function) that was the inverse of the used frequency. That is, the weighting matrix was a diagonal where $\mathbf{W}(i,i) = 1/\omega_i$. Or they used some fixed integers such as $\mathbf{W} = \text{diag}(10,\cdots,10,.\ 1,\cdots,1,\cdots)$. In [1], there were no guidelines or tools to choose an appropriate scale factors (or functions) relating to the data.

We propose a kind of method to compute a weighting function systematically. The relationship between the obtained model and the reality they represent is subtle and complex. The quality of a model depends on how closely its responses match those of the true plant. In our case, because the number of measurements is limited, and the adopted method for identification is quite poor when compared to the special identification algorithms such as ARMAX etc.[5], the quality of our model must be inferior as well. Hassul's method, however, is good enough for the use of undergraduate students if there is an effective method to find a weighting function dependent on the data. For this purpose, we propose the following procedure to improve

the quality of the model when using Hassul's method.

Step 1: Divide the frequency region of the plant into several smaller parts.

To Improve the accuracy of measurements, we divide the frequency region into several (say, k times) smaller parts. Then the measurement is carried out laying emphasis on one part at a time. By iterating this process k times, we could obtain a high quality of experimental data.

Step 2: Identifying the model by Eq.(11)

By repeating the Hassul's method k times and taking an average of them, we can grasp the approximate transfer function $G(s)$ of the given plant that must have some errors especially in the low frequency region. We regard it as a nominal model $G_{nom}(s)$.

Step3: Determine a weighting function

It is well known there are two types of uncertainty related to the identified model, that is, additive and multiplicative uncertainty [6]:

$$\begin{aligned} \tilde{G}(s) &= G_{nom}(s) + \Delta W_a(s) \\ \tilde{G}(s) &= (1 + \Delta W_m(s)) G_{nom}(s) \end{aligned} \qquad (14)$$

where $\tilde{G}(s)$ means the set of models where there must be a real one. In our case, the additive uncertainty seems to be more suitable because it shows the real difference in the frequency region. From Eq.(14), we obtain

$$\Delta W_a(s) = \tilde{G}(s) - G_{nom}(s) \qquad (15)$$

As a $\tilde{G}(s)$, we use k transfer functions which were already determined in the step 2. By substituting $\tilde{G}(s)$ with k number of transfer functions, we can compute $W_a(s)$ which will be used as a weighting function in Eq.(11).

The above method seems quite reasonable because the differences dependent on frequency were taken into consideration. We will next use examples to prove this even further.

4. Examples

Here, we present three examples to verify the validity of the proposed method. In order to compare with the results of Hassul's, we must first employ the same data which was used in [1].

We start with the frequency data point shown with open circles in Fig.1 and 2. The monotonic gain plot, along with the final slope or phase, indicates that this transfer function can be modeled as a third-order all-pole system. So, we chose n=3 and m=0. The Bode diagram for this transfer function by Hassul's and the proposed method was shown in dotted and solid lines in Fig.1, respectively. And the dash-dot line in the figure represents the result without the weighting function. It is clear from the figure that the proposed method made the fits in the low frequency region better.

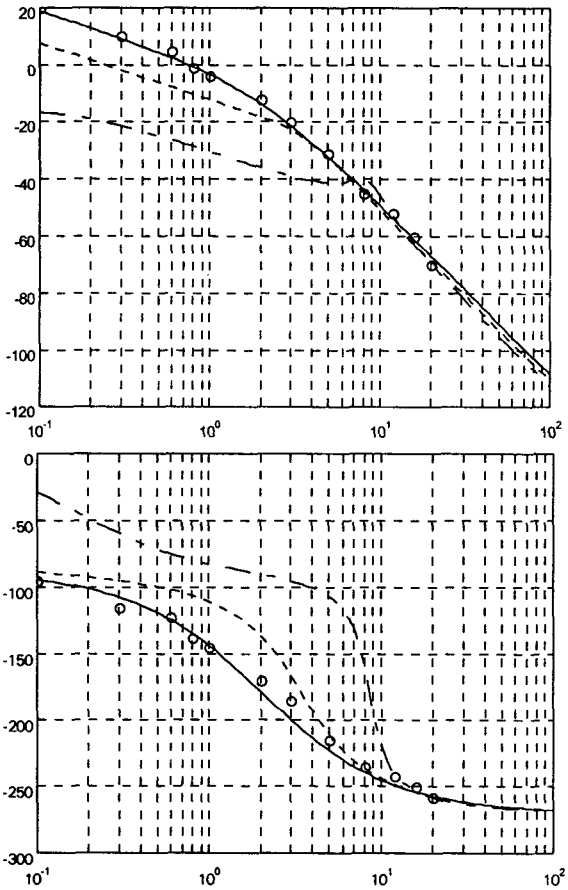


Fig.1 Third-order all-pole data, top: Magnitude. Bottom: Phase. Open circles: Experimental data

Fig.2 shows the identified result of the second example. The transfer function chosen to fit this data has a sixth-order numerator and seventh-order denominator. From the figure, we see that the method of this paper gives a better fit in the low frequency region. This example also implies that, if we really want to obtain more of an accurate model, the special identification algorithms must be applied. In that case, a lot of data and mathematical understanding is indispensable.

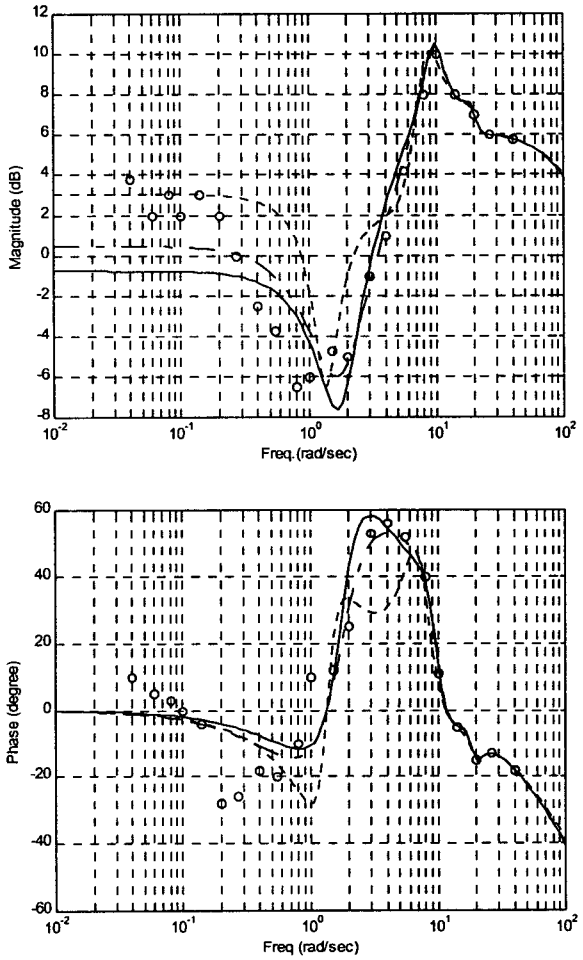


Fig.2 Example 2: Top: Magnitude. Bottom: Phase. Open circles: Experimental data

In the third example, a third-order chebyshev type filter was adopted. That is, the transfer function of the designed filter is:

$$G(s) = \frac{309673}{s^3 + 347s^2 + 8987s + 317356}$$

We realized this, and then impressed sine function as an input and measured the output response (magnitude and phase) by making use of the steps in the previous section. At this time, we divided the frequency region into three parts. Fig.3 represents the frequency plot (that is, additive uncertainty) to determine the weighting function, by which we obtained $W(s)$ as

$$W_a(s) = \frac{609.2s + 6331.7}{s^3 + 15.67s^2 + 264.7s + 837.37} \qquad (16)$$

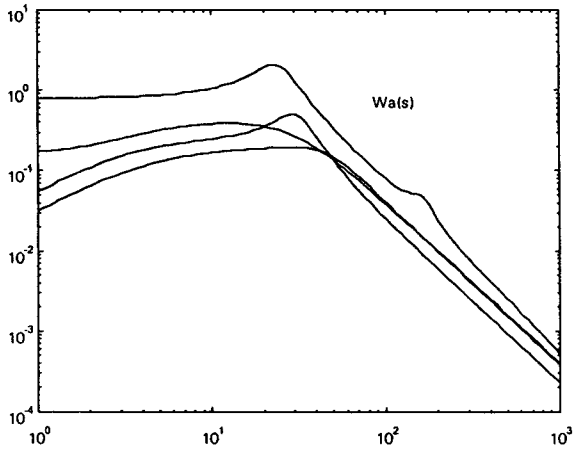
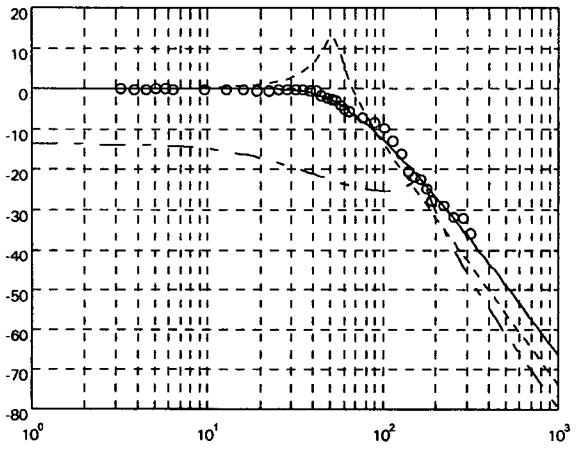
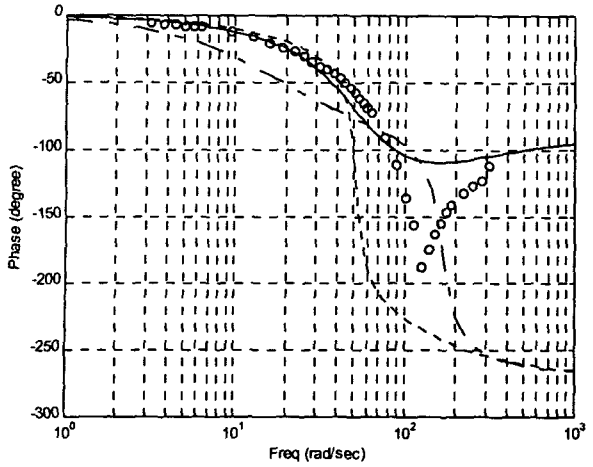


Fig.3 Determination of Weighting function

The Bode diagram of the identified transfer function was shown in Fig.4. We know that the proposed method also works quite well when compared to Hassuls's result.



(a) Magnitude



(b) Phase

Fig.4 The result of third-order filter

5. Conclusion

We focused on the system identification for undergraduate students. The usual identification process taken in the undergraduate course is limited to straight-line asymptotic fits to experimental magnitude data. Although this technique is quite helpful in the case of simple plants (for example, a plant has simple poles or zeros), generally it is very difficult to estimate the transfer function only by using this classical method. Hassul suggested that undergraduate students should be exposed to an automatic transfer function synthesis. However, they didn't mention how to determine a weighting function(they said "scale factor") which plays a crucial role in applying their algorithm to real problems.

We proposed one simple method to obtain a weighting function for improving the quality of the model especially in the low frequency region, and its validity was certified through some examples. Because the proposed weighting function is based on the model uncertainty, it is likely to be useful when one wants to design a robust controller for the given plant.

6. References

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