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A Nonlinear Analytic Function Expansion Nodal Method for Transient Calculations

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Abstract

The nonlinear analytic function expansion nodal (AFEN) method is applied to the solution of the time-dependent neutron diffusion equation. Since the AFEN method requires both the particular solution and the homogeneous solution to the transient fixed source problem, the derivation of the solution method is focused on finding the particular solution efficiently. To avoid complicated particular solutions, the source distribution is approximated by quadratic polynomials and the transient source is constructed such that the error due to the quadratic approximation is minimized. In addition, this paper presents a new two-node solution scheme that is derived by imposing the constraint of current continuity at the interface corner points. The method is verified through a series of applications to the NEACRP PWR rod ejection benchmark problems.

I. Introduction

It has been well verified that the Analytic Function Expansion Nodal (AFEN)[1] method provides superior accuracy in the eigenvalue calculations for steady reactor states. The superior accuracy of the AFEN method originates from the explicit representation of the two-dimensional, intranodal neutron flux distribution during the course of solving the nodal neutron balance equation. It is distinguished from the transverse-integrated nodal methods that employ only one-dimensional flux representation in each direction. Recently, the nonlinear iteration technique was applied to the AFEN method to provide an alternative acceleration scheme.[2] The application of the nonlinear AFEN method, however, was limited to the steady-state eigenvalue problems. Since one of the primary benefits of the nonlinear iteration technique could lie in the possibility of infrequent nodal calculations during a transient calculation, which can save the transient calculation time significantly, there is a strong motivation for developing a nonlinear AFEN method for transient calculations.

In the transient calculation, the problem to be solved is a fixed source problem that is formulated by the temporal discretization of the time derivative term appearing in the time-dependent neutron diffusion equation. The presence of the independent source poses a new problem in the AFEN solution process, which is not encountered during the solution of eigenvalue problems. That is the need for finding the particular solution for the source distribution formed at each time point depending on the intranodal flux distribution at the previous

time point. In principle, the true particular solution corresponding to this source becomes trigonometric and/or hyperbolic functions. Obtaining the analytic solution containing the true particular solution would be very complicated and makes the solution process inefficient. In the work here, the source distribution is approximated by simple quadratic polynomials so that the nonlinear AFEN method is readily applicable to the transient calculation. The error associated with the quadratic source approximation is minimized by reorganizing the transient neutron balance equation such that the resulting equation has less influence on the flux induced from the source.

In the following section, a two-node AFEN solution method is derived for a fixed source problem. The derivation is distinguished from the existing derivation[2] in that an additional constraint of continuity in the directional current is imposed at the corner points shared by the two nodes. In the derivation, the source distribution is assumed to be quadratic in both x and y directions. Section III then describes the method for constructing the quadratic source distribution. Node average values of the eight surrounding nodes as well as the node itself are used in the process. Section IV presents the results of the applications of the method to the calculation of a set of NEACRP PWR rod ejection benchmark problems.[3] Finally, conclusion are drawn in Section V.

II. Two-Node AFEN Solution for a Fixed Source Problem

In the nodal update step during the nonlinear iteration process, a two-node problem should be solved for every nodal interface. The neutron balance equation to be solved in the two-node problem is derived by integrating the two-group, three-dimensional neutron balance equation for a fixed source problem over the axial direction and it is given as:

$$-D\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+A\phi=S_t(x,y)-L_z(x,y)\equiv S(x,y) \tag{1}$$

where

$$D\equiv\begin{bmatrix}D_1 & \\ & D_2\end{bmatrix},\quad A=\begin{bmatrix}\Sigma_{r1}-\lambda\nu\Sigma_{f1} & -\lambda\nu\Sigma_{f2}\\ -\Sigma_{12} & \Sigma_{r2}\end{bmatrix},\quad \lambda=\frac{1}{k_{eff}} \tag{2}$$

and

$$L_z(x,y)\equiv\begin{bmatrix}L_{z1}\\ L_{z2}\end{bmatrix},\quad L_{zg}\equiv\frac{1}{h_z}\left(J^+_{zg}(x,y)-J^-_{zg}(x,y)\right). \tag{3}$$

Note that the flux (ϕ) and transient source vectors (S_t) are two-element vectors whose elements are defined for each group. Eq. (1) is solved below based on the assumption that the distribution of the source that combines the transient source and the axial leakage can be approximated by a quadratic polynomial of the following form:

$$S(x,y)=\sum_{i=0}^{2}\sum_{j=0}^{2}b_{ij}f_i(\xi_x)f_j(\xi_y) \tag{4}$$

where

$$f_0(\xi)=1,\quad f_1(\xi)=\xi,\quad f_2(\xi)=3\xi^2-\frac{1}{4},\quad \textit{and}\quad \xi_u=\frac{u}{h_u}. \tag{5}$$

Determination of the coefficients b_{ij} is discussed in the next section.

Eq. (1) can be solved analytically in the interior of each node and the analytic solution consists of the homogeneous and particular solutions, i.e.,

$$\phi(x,y) = \phi^H(x,y) + \phi^P(x,y) \tag{6}$$

Employing the AFEN method, the homogeneous solution is obtained as:

$$\phi^H(x,y) = \begin{bmatrix} \phi_1^H(x,y) \\ \phi_2^H(x,y) \end{bmatrix} = \begin{bmatrix} 1 & s \\ r & 1 \end{bmatrix} \begin{bmatrix} \varphi_0(x,y) \\ \varphi_1(x,y) \end{bmatrix} \tag{7}$$

where the modal fluxes are defined as:

$$\begin{aligned} \varphi_m(x,y) &= a_{m0} cn_m(B_m x) + a_{m1} sn_m(B_m x) + a_{m2} cn_m(B_m x) + a_{m3} sn_m(B_m x) \\ &+ a_{m4} cn_m\left(\frac{B_m x}{\sqrt{2}}\right) cn_m\left(\frac{B_m y}{\sqrt{2}}\right) + a_{m5} sn_m\left(\frac{B_m x}{\sqrt{2}}\right) cn_m\left(\frac{B_m y}{\sqrt{2}}\right) \\ &+ a_{m6} sn_m\left(\frac{B_m x}{\sqrt{2}}\right) sn_m\left(\frac{B_m y}{\sqrt{2}}\right) + a_{m7} cn_m\left(\frac{B_m x}{\sqrt{2}}\right) sn_m\left(\frac{B_m y}{\sqrt{2}}\right) \end{aligned} \tag{8}$$

with the generic functions, cn_m and sn_m, which are *cosh* and *sinh* functions, respectively, unless $m=0$ and $k_\infty > k_{eff}$ in which case they become, respectively, *cos* and *sin* functions. The r and s factors appearing in the transformation matrix in Eq. (7) are, respectively, the thermal-to-fast flux ratio corresponding to the fundamental mode buckling (B_0^2) and the fast-to-thermal flux ratio for the first harmonics mode buckling (B_1^2).

For the right hand side (RHS) given by a quadratic function of Eq. (4), the particular solution of Eq. (1) can be obtained as:

$$\phi^P(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{ij} f_i(\xi_x) f_j(\xi_y) \tag{9}$$

The coefficients of the particular solution can be determined easily by inserting Eq. (9) into Eq. (1) and by requiring the coefficients of the same order functions appearing on both sides to be the same. The results are as follows:

$$c_{ij} = \begin{cases} A^{-1} b_{ij} & for \quad (i,j) \in \{(1,1),(2,1),(1,2),(2,2)\} \\ A^{-1}\left(b_{ij} - \dfrac{6D}{h_x^2} c_{i'j'}\right) & for \quad (i,j) \in \{(0,1),(0,2),(1,0),(2,0)\} \\ A^{-1}\left(b_{ij} - \dfrac{6D}{h_x^2}\left(c_{ij+2} + c_{i+2j}\right)\right) & for \quad (i,j) = (0,0) \end{cases} \tag{10}$$

On the second line above $i' = i$ *and* $j' = j+2$ if $j = 0$ and vice versa. Note that these coefficients are uniquely determined at each node given b_{ij} ' s, without any constraints applied.

With the particular solution determined, there are sixteen homogeneous solution coefficients (8/mode × 2 modes) to be determined to finalize the solution at each node. In order to determine these coefficients uniquely in a two-node problem, thirty-two constraints are needed since there are two sets of coefficients in the two-node problem. The constraints provided per group in Ref. 2 were two node-average fluxes, six corner fluxes, four surface-average transverse direction currents, two corner flux continuities (at the top and bottom corners on the interface), and finally the continuity of surface averaged fluxes and currents. In the present derivation, however,

the two corner fluxes at the interface are not provided, but instead the corner current continuity condition is imposed in the x-direction at the two corners located on the interface. The reason for this change is to eliminate the additional calculation step needed to compute the corner fluxes accurately when the former scheme is employed. The additional step is to determine the solution of a block penta diagonal linear system for corner fluxes, which is obtained by imposing the constraint of so called the corner point balance (CPB).[1] The computational overhead of the additional step could be eliminated by the use of the method of successive smoothing (MSS) which is based on the assumption of linear flux variation near the corner points.[2] However, the penalty of the cruder evaluation of the corner fluxes by MSS appears in the accuracy of the solution.

In contrast to the CPB or MSS based corner flux evaluation schemes, the present scheme determines the corner fluxes at the interface during the two-node solution. One problem, however, is that there are four estimates of a corner flux because there are four two-node solutions obtained around a corner point at the end of all the two-node calculations. This problem is overcome by taking the arithmetic average of the four estimates and then by defining the averaged value as the true corner flux. Then it is possible to functionalize the corner flux as a linear combination of the node average fluxes of the four surrounding nodes with the linear combination coefficients being determined by the results of the four two-node calculations. Once the corner point flux is represented as a function of node average fluxes, the new corner fluxes after a coarse mesh finite difference (CMFD) calculation are calculated readily using the newly determined node average flux distribution.

When imposing the constraints, note that the contribution from the particular solution to the constraints should be incorporated properly. For instance, the node average flux constraint on the homogenous solution becomes:

$$\bar{\phi}_i^H = \bar{\phi}_i - \bar{\phi}_i^P \quad for \quad i \in \{l, r\} \quad . \tag{11}$$

By imposing the thirty-two constraints, a 32x32 linear system is formulated for the thirty-two unknown coefficients. Since the 32x32 matrix is sparse and have only a few distinct entries, the linear system can be readily reduced algebraically to a 6x6 linear system containing the surface averaged currents and two corner fluxes as unknowns. The interface currents (one for each group) determined by solving the 6x6 linear system are then used to update the correctional nodal coupling coefficient for use in the subsequent CMFD calculation.

III. Formulation of Transient Fixed Source

The transient fixed source problem is formulated by applying a temporal differencing scheme to the following time-dependent neutron diffusion equation:

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = (F_p - M)\phi + S_d \quad . \tag{12}$$

The prompt fission operator in the above equation can be represented by $F_p = (I - \beta)F$ where β is a matrix representing node-wise total effective delayed neutron fractions and F is the operator for the total fission neutron source that includes prompt and delayed neutrons.

There are several methods to discretize Eq. (12) in time. Typical examples are the fully-implicit method, the

Crank-Nicholsen method (the θ method with θ being 0.5), the frequency transformation method, and so on. In the following presentation, the fully-implicit method is used for simplicity. The presentation is, however, applicable to all the other temporal discretization schemes. Applying the Euler method to Eq. (12) at time point *n* yields:

$$\left(\frac{I}{v\Delta t}+M_n\right)\phi_n-\left((I-\beta)F_n\phi_n+S_d^n\right)=\frac{I}{v\Delta t}\phi_{n-1} \quad . \tag{13}$$

It can be shown[4] that the delayed neutron source at t_n (S_d^n) in the above equation can be represented by the fission source and a known delayed neutron source terms as $S_d^n=\omega F_n\phi_n+\tilde{S}_d^{n-1}$ so that Eq. (13) be transformed into:

$$\left(\frac{I}{v\Delta t}+M_n-\beta_p F_n\right)\phi_n=\frac{I}{v\Delta t}\phi_{n-1}+\tilde{S}_d^{n-1} \tag{14}$$

where $\beta_p=1-\beta+\omega$. Eq. (14) is the true transient fixed source problem to be solved during the transient calculation. In principle, the intranodal distribution of the source appearing on the RHS of the above equation can be obtained by using the intranodal flux distribution obtained at the previous time point. The source distribution corresponding to the AFEN solution then consists of the sixteen terms of trigonometric and/or hyperbolic functions. Since retaining the full source distribution is impractical in that it requires enormous additional memory as well as in that the solution process of finding the particular solution becomes very complicated, the source distribution here is approximated by a quadratic polynomial as given in Eq. (4). If the source corresponding to the RHS of Eq. (14) is approximated by a quadratic polynomial, however, the error due to the approximation may be significant because the absolute magnitude of the source is large. On the other hand, since the operator on the LHS is different from the steady-state one, nonphysical perturbations could be introduced so that even the solution of a null transient could not preserve the steady-state solution.

In these regards, the operator on the LHS is changed such that it is identical to the steady-state one and this change yields a pseudo transient problem given as:

$$(M_n-F_n)\phi_n=\tilde{S}_d^{n-1}-\frac{I}{v\Delta t}(\phi_n-\phi_{n-1})-(\beta-\omega)F_n\phi_n\equiv S_t^n \quad . \tag{15}$$

Note that this transform of the transient fixed source problem eliminates both the large source and the null transient problems. It is necessary, however, to update the pseudo transient source (S_t^n) iteratively using the most recent values of the node average fluxes because it contains the fluxes at the current time point (ϕ_n). This approach was originally introduced by Engrand and *et. al.*[4] and was implemented in a group of spatial kinetics codes such as PARCS[5].

The quadratic polynomial in two-dimensions, given by Eq. (4), contains nine coefficients. They can be determined uniquely by using nine node-average effective source values out of which eight values come from the eight nodes surrounding the node of interest. Note that the effective source is obtained by combining the axial leakage and the transient source. An alternative to obtaining the quadratic polynomial is to use the source values at the four corners and the four surface values instead of those of the eight surrounding nodes, if they are readily available. A crude approximation of the source distribution is to use a constant at each node or more simply not to use the source at all. The impact of these cruder approximations are investigated in the next section

IV. Verification

For the sake of verification of the transient nonlinear AFEN method described in the previous sections, a series of benchmark calculations for the NEACRP PWR control rod ejection problem[3] was performed using the PARCS code[5] to which the nonlinear AFEN method is added as an additional calculation option. The basic nodal method of the PARCS code is the nonlinear analytic nodal method (ANM) that is one of the transverse-integrated nodal methods. The nonlinear ANM provides very accurate solutions to these benchmark problems as long as four radial nodes are used per fuel assembly (FA). With one-node/FA, however, the accuracy of the transient results deteriorates so that the errors in the core power change are not negligible. Hence the focus of this verification was on the accuracy of the one-node/FA AFEN solution.

Among the six benchmark problems, results are presented for the two most difficult problems, Cases A1 and C1, that involve a single rod ejection at the center and at the periphery of the core, respectively, from a initially heavily rodded HZP state. Table I summarizes the eigenvalue calculation results for the initial steady state. It compares the errors in the eigenvalue and in the radial power distribution for different radial node sizes and for both the ANM and AFEN options. The reference power distribution was taken from a sixteen-nodes/FA ANM solution and the boron concentration used were 560.93 and 1128.04 ppm for Cases A1 and C1, respectively. These are the critical concentrations for the sixteen-nodes/FA cases. Note that these values are very close to the published reference values obtained by the PANTHER code[6] which are 561.20 and 1128.29 ppm, respectively. As shown in Table I, the errors in k_{eff} as well as in the power distribution are all negligible except for the one-node/FA ANM cases. The corresponding AFEN cases, however, show remarkable improvements over the ANM for both Cases A1 and C1. This comparison demonstrates that the new two-node AFEN solution scheme derived in Section II provides very accurate solutions even in the eigenvalue calculations.

The transient calculation results are shown in Figures 1 and 2. The upper part of each figure compares the transient core power changes obtained by using both ANM and AFEN with one and four-nodes/FA, respectively, together with the reference PANTHER solutions. On the other hand, the lower part compares three different source treatment schemes employed in the two-node AFEN calculation for the one-node/FA configuration: no source, constant source within each node, and the quadratic source distributions represented by Eq. (4). In both lower parts, it is clearly demonstrated that the source treatment scheme affects the transient response significantly and the use of the quadratic source distribution is essential. In the upper part of Figure 1, it is shown that the four-nodes/FA ANM as well as AFEN solutions agree well with the reference, but the one-node/FA ANM solution is very much off from the reference. The corresponding AFEN solution is much more accurate than the ANM solution even though it is slightly worse than the four-nodes/FA solutions. For Case C1, the improvement of the solution attained by the AFEN method for the one-node/FA case is marginal since the corresponding ANM solution is not as bad as in Case A1. From these comparisons, it can be drawn that the one-node/FA AFEN solutions are reasonably good as long as the quadratic source treatment scheme is employed

V. Conclusions

A nonlinear AFEN method for efficient transient calculations was developed based on a quadratic approximation of the intranodal source distribution and a new two-node AFEN solution scheme requiring current continuity at the interface corner points. The new two-node AFEN solution scheme turned out to be very accurate as shown in the eigenvalue calculations presented in Table I. The transient AFEN solution method provides more accurate solutions than the ANM especially when one-node/FA is used. The slight error in the transient core power response noted in the HZP rod ejection calculations might be reduced further if the source distribution and consequently the particular solution is represented by higher order functions.

References

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Table I. Comparison of Solution Accuracy for the Initial State Calculations of the Selected NEACRP Problems

Nodes per FA	Option	Case A1				Case C1			
		k_{eff} Error pcm	RPD Error, %			k_{eff} Error pcm	RPD Error, %		
			RMS	Max. Neg.	Max. Pos.		RMS	Max. Neg.	Max. Pos.
4	ANM	3.2	0.18	-0.16	0.42	4.7	0.10	-0.19	0.19
	AFEN	-6.1	0.19	-0.49	0.20	-6.0	0.14	-0.45	0.20
1	ANM	42.8	2.14	-2.03	4.27	67.3	1.30	-2.26	2.57
	AFEN	-3.1	0.57	-1.41	1.22	7.1	0.33	-0.60	0.76

Figure 1. Core Power Changes for NEACRP A1 Obtained by Using Various Options

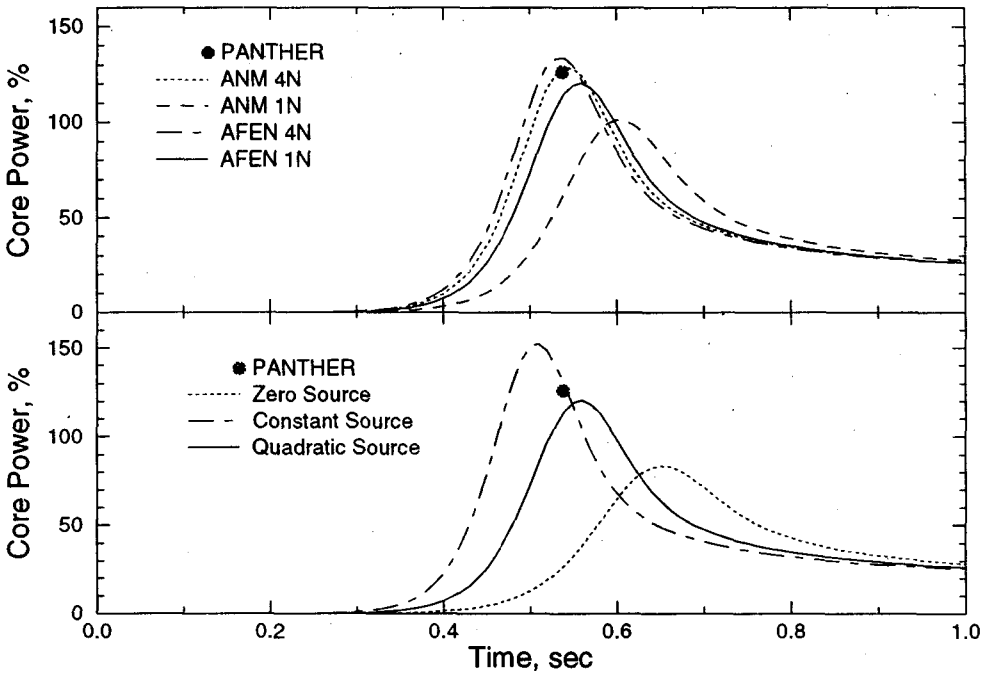

Figure 2. Core Power Changes for NEACRP C1 Obtained by Using Various Options